Text Book for

INTERMEDIATE

Second Year

Mathematics

Paper - IIA

Algebra, Probability

Telugu and Sanskrit Akademi

Andhra Pradesh

Intermediate
Second Year

Mathematics
Paper - IIA
Text Book

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Y.S. JAGAN MOHAN REDDY

CHIEF MINISTER
ANDHRA PRADESH

AMARAVATI

MESSAGE

I congratulate Akademi for starting its activities with printing of textbooks from the academic year 2021 – 22.

Education is a real asset which cannot be stolen by anyone and it is the foundation on which children build their future. As the world has become a global village, children will have to compete with the world as they grow up. For this there is every need for good books and good education.

Our government has brought in many changes in the education system and more are to come. The government has been taking care to provide education to the poor and needy through various measures, like developing infrastructure, upgrading the skills of teachers, providing incentives to the children and parents to pursue education. Nutritious mid-day meal and converting Anganwadis into pre-primary schools with English as medium of instruction are the steps taken to initiate children into education from a young age. Besides introducing CBSE syllabus and Telugu as a compulsory subject, the government has taken up numerous innovative programmes.

The revival of the Akademi also took place during the tenure of our government as it was neglected after the State was bifurcated. The Akademi, which was started on August 6, 1968 in the undivided state of Andhra Pradesh, was printing text books, works of popular writers and books for competitive exams and personality development.

Our government has decided to make available all kinds of books required for students and employees through Akademi, with headquarters at Tirupati.

I extend my best wishes to the Akademi and hope it will regain its past glory.

(Y.S. JAGAN MOHAN REDDY)

Dr. NANDAMURI LAKSHMIPARVATHi
M.A., M.Phil., Ph.D.
Chairperson, (Cabinet Minister Rank)
Telugu and Sanskrit Akademi, A.P.

Message of Chairperson, Telugu and Sanskrit Akademi, A.P.

In accordance with the syllabus developed by the Board of Intermediate, State Council for Higher Education, SCERT etc., we design high quality Text books by recruiting efficient Professors, department heads and faculty members from various Universities and Colleges as writers and editors. We are taking steps to print the required number of these books in a timely manner and distribute through the Akademi's Regional Centers present across the Andhra Pradesh.

In addition to text books, we strive to keep monographs, dictionaries, dialect texts, question banks, contact texts, popular texts, essays, linguistics texts, school level dictionaries, glossaries, etc., updated and printed and made available to students from time to time.

For competitive examinations conducted by the Andhra Pradesh Public Service Commission and for Entrance examinations conducted by various Universities, the contents of the Akademi publications are taken as standard. So, I want all the students and Employees to make use of Akademi books of high standards for their golden future.

Congratulations and best wishes to all of you.

N. Lakshmiparvathi

(NANDAMURI LAKSHMIPARVATHI)

J. SYAMALA RAO, I.A.S.,
Principal Secretary to Government

Higher Education Department
Government of Andhra Pradesh

MESSAGE

I Congratulate Telugu and Sanskrit Akademi for taking up the initiative of printing and distributing textbooks in both Telugu and English media within a short span of establishing Telugu and Sanskrit Akademi.

Number of students of Andhra Pradesh are competing of National Level for admissions into Medicine and Engineering courses. In order to help these students Telugu and Sanskrit Akademi consultation with NCERT redesigned their Textbooks to suit the requirement of National Level Examinations in a lucid language.

As the content in Telugu and Sanskrit Akademi books is highly informative and authentic, printed in multi-color on high quality paper and will be made available to the students in a time bound manner. I hope all the students in Andhra Pradesh will utilize the Akademi textbooks for better understanding of the subjects to compete of state and national levels.

(J. SYAMALA RAO)

THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC] and to secure to all its citizens:

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the [unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.

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Foreword

The Government of India vowed to remove the educational disparities and adopt a common core curriculum across the country especially at the Intermediate level. Ever since the Government of Andhra Pradesh and the Board of Intermediate Education (BIE) swung into action with the task of evolving a revised syllabus in all the Science subjects on par with that of CBSE, approved by NCERT, its chief intention being enabling the students from Andhra Pradesh to prepare for the National Level Common Entrance tests like NEET, ISEET etc for admission into Institutions of professional courses in our Country.

For the first time BIE AP has decided to prepare the Science textbooks. Accordingly an Academic Review Committee was constituted with the Commissioner of Intermediate Education, AP as Chairman and the Secretary, BIE AP; the Director SCERT and the Director Telugu Akademi as members. The National and State Level Educational luminaries were involved in the textbook preparation, who did it with meticulous care. The textbooks are printed on the lines of NCERT maintaining National Level Standards.

The Education Department of Government of Andhra Pradesh has taken a decision to publish and to supply all the text books with free of cost for the students of all Government and Aided Junior Colleges of newly formed state of Andhra Pradesh.

We express our sincere gratitude to the Director, NCERT for according permission to adopt its syllabi and curriculum of Science textbooks. We have been permitted to make use of their textbooks which will be of great advantage to our student community. I also express my gratitude to the Chairman, BIE and the honorable Minister for HRD and Vice Chairman, BIE and Secretary (SE) for their dedicated sincere guidance and help.

I sincerely hope that the assorted methods of innovation that are adopted in the preparation of these textbooks will be of great help and guidance to the students.

I wholeheartedly appreciate the sincere endeavors of the Textbook Development Committee which has accomplished this noble task.

Constructive suggestions are solicited for the improvement of this textbook from the students, teachers and general public in the subjects concerned so that next edition will be revised duly incorporating these suggestions.

It is very much commendable that Intermediate text books are being printed for the first time by the Akademi from the 2021-22 academic year.

Sri. V. Ramakrishna I.R.S.
Director
Telugu and Sanskrit Akademi,
Andhra Pradesh

Preface

The Board of Intermediate Education, has recently revised the syllabus in Mathematics for the Intermediate Course with effect from the Akademic year 2012-13. Accordingly, Telugu Akademi has prepared the necessary Text Books in Mathematics.

In accordance with the current syllabus, the topics relating to paper II-A : **Algebra** and **Probability** are dealt with in this book. The syllabus is presented in ten chapters. Algebra part given in seven chapters : **Complex numbers, De Moivre's Theorem, Quadratic Expressions, Theory of Equations, Permutations and Combinations, Binomial Theorem, Partial Fractions** and Probability part given in three chapters **Measures of Dispersion, Probability and Random Variables and Probability Distributions.**

Further, for the benfit of students intending to appear for All India Level Competitive Examinations, the **Additional Reading Material** is included in the Appendix, It contains two chapters Exponential and Logarithmic Series and Linear Programming. These topics ar for additional reading, but not for examinations. **No question will be set on the Additional Reading Material, in the Intermediate II Year Public Examination, Mathematics, paper- IIA.**

Every chapter herein. is divided into various sections and subsections. depending on the contents discussed. These contents are strictly in accordance with the prescribed syllabus and they reflect faithfully the scope and spirit of the same. Necessary definitions, theorems, corollaries. proofs and notes are given in detail. Key concepts are given at the end of each chapter, Illustrative examples and solved problems are in plenty. and these shall help the students in understanding the subject matter.

Every chapter contains exercises in a graded manner which enable the students to solve them by applying the knowledge acquired. All these problems are classified according to the nature of their answers as **I - very short II short and III-long.** Answers are provided for all the exercises at the end of each chapter.

Keeping in view the National level competitive examinations, some concepts and notions are highlighled for the benefit of the students. Care has been taken regarding rigor and logical conSistency in the presenlation of concepts and in proving theorems. Alt the end of the text Book, a lisl of some **Reference Books** in the subject malter is furnished.

The Members of the Mathematics Subject Committee, constituted by Board of Intermediate Education, were invited to interact with the team of the Authors and Editors. They pursued the contenls chapter wise and gave some useful suggestions and comments which are duly incorporated. The special feature oflhis Book, brought out in a new formal, is that each chapter begins with a thought mostly on Mathematics. through a quotation from a famous thinker. It carries a portrait of a noted : mathematician with a brief write-up.

In the concluding part of each chapter some relevant historical notes are appended. Wherever found appropriate, references are also made of the contributions of ancient Indian scientists to the advancement of Methamatics. The purpose is to enable the students to have a glimpse into the history of Mathematics in general and the oontributions of Indian methematicians in particular.

Inspite of enough care taken in the scrutiny at various stages in the preparation of the book, errors might have crept in. The readers are therefore, requested to identify and bring them to the notice of the Akademi. We will appreciate if suggetions to enhance the quality of the book are given. Efforts will be made to incorporate them in the subsequent editions.

Prof. P.V. Arunachalam
Chief Coordinator

Preface to the Reviewed Edition

Telugu Akademi is publishing Text books for Two year Intermediate in English and Telugu medium since its inception, periodical review and revision of these publications has been undertaken as and when there was an updation of Intermediate syllabus.

In this reviewed Edition, now being undertaken by the Telugu Akademi, Andhra Pradesh the basic content of its earlier Edition is considered and it is reviewed by a team of experienced teachers. Modification by way of correcting errors, print mistakes, incorporating additional content where necessary to elucidate a concept and / or a definition, modification of existing content to remove obscurities for releasing the concept more easily are carried out mainly in this review.

Not withstanding the effort and time spent by the review team in this endeavour, still a few aspects that still need modification or change might have been left unnoticed.

Constructive suggestions from the academic fraternity are welcome and the Akademi will take necessary steps to incorporate them in the forth coming edition.

We appreciate the encouragement and support extended by the Academic and Administrative staff of the Telugu Akademi in fulfilling our assignment with satisfaction.

Editors
(Reviewed Edition)

Contents

ADDITIONAL READING MATERIAL

For the benefit of students who want to appear for competitive exams based on COBSE the following topics may be given as Additional Reading Material.

Algebra

Chapter 1

Complex Numbers

"Mathematics is the Queen of Sciences and Number theory is the Queen of Mathematics"

*– **Gauss***

Introduction

In the earlier classes we have learnt the properties of real numbers and studied certain operations on real numbers like addition, subtraction, multiplication and division. We have also learnt solving linear equations in one and two variables and quadratic equations in one variable. We have seen that the equation $x^2 + 1 = 0$ has no real solution since the square of every real number is non–negative. This suggests that we need to extend the real number system to a larger system, so that we can account for the solutions of the equation $x^2 = -1$. If this is done, it would help solving the equation $ax^2 + bx + c = 0$ for the case $b^2 - 4ac < 0$, which is not possible in the real number system.

It was Euler (1707 – 1783) that identified a root for the quadratic equation $x^2 + 1 = 0$ with the symbol i.

Hamilton

(1805 – 1865)

William Rowan Hamilton *was an Irish mathematician, physicist, and astronomer who made important contributions to the development of optics, dynamics and algebra. His discovery of quaternions is perhaps his best known investigation. Dr. Brinkley remarked of Hamilton at the age of eighteen: "This young man, I do not say will be, but is, the first mathematician of his age."*

Euler called it an imaginary root and termed the numbers of the form $a + ib$, $a, b \in \mathbf{R}$ as complex numbers. However, we define a complex number in a more general way – following the definition given by Hamilton (1805 – 1865) as an ordered pair of real numbers (a, b) and study their representation, equality and some algebraic operations on the set of complex numbers. We shall also discuss some concepts like conjugacy, modulus and amplitude of a complex number. Finally we give a geometrical representation of a complex number and introduce the Argand plane and Argand diagram to make out the sum, difference, product and quotient of two complex numbers geometrically.

1.1 Complex number as an ordered pair of real numbers Fundamental operations

In this section, we shall give a general definition of a complex number and introduce certain (algebraic) operations on the set of complex numbers to develop the algebraic structure of complex numbers.

1.1.1 Definition (Complex number)

A complex number is an ordered pair of real numbers (a, b). *The set of all complex numbers is denoted by* $\mathbf{C} = \{(a, b) \mid a \in \mathbf{R}, b \in \mathbf{R}\} = \mathbf{R} \times \mathbf{R}$.

1.1.2 Definition (Equality of complex numbers)

Two complex numbers $z_1 = (a, b)$ *and* $z_2 = (c, d)$ *are said to be equal if* $a = c$ *and* $b = d$.

1.1.3 Definitions

(i) **Addition of complex numbers**

If $z_1 = (a, b)$ and $z_2 = (c, d)$, we define

$z_1 + z_2 = (a, b) + (c, d)$ to be the complex number $(a + c, \ b + d)$.

(ii) **Negative of a complex number**

The negative of any complex number $z = (a, b)$ denoted by $-z$ is defined as $-z = (-a, -b)$.

(iii) **Difference of complex numbers**

If $z_1 = (a, b)$ and $z_2 = (c, d)$, then we define $z_1 - z_2 = z_1 + (-z_2)$.

Then $z_1 - z_2 = (a - c, \ b - d)$.

(iv) **Multiplication of complex numbers**

If $z_1 = (a, b),\ z_2 = (c, d)$ then we define their product by

$z_1 . z_2 = (a,\ b) . (c,\ d) = (ac - bd,\ ad + bc)$

1.1.4 Examples

If $z_1 = (2, 3)$ and $z_2 = (-6, 5)$, then $z_1 + z_2 = (2 - 6,\ 3 + 5) = (-4,\ 8)$

If $z_1 = (6, 3)$ and $z_2 = (2, -1)$, then $z_1 - z_2 = (6 - 2, 3 - (-1)) = (4, 4)$

If $z_1 = (1, 2)$ and $z_2 = (3, -4)$, then $z_1 . z_2 = (1, 2) . (3, -4) = (3 + 8, -4 + 6) = (11,\ 2)$

Note: From the definition of addition of two complex numbers, it is clear that

$\alpha \in \mathbf{C},\ \beta \in \mathbf{C} \ . \Rightarrow \alpha + \beta \in \mathbf{C}$

In a similar manner $\alpha . \beta \in \mathbf{C}$. Hence the operations of addition and multiplication are both binary operations on $\mathbf{C}$. We denote $\alpha . \beta$ by $\alpha\beta$ also.

We now prove the following laws of addition on $\mathbf{C}$.

1.1.5 Theorem

(i) *In* $\mathbf{C}$, *addition is associative*

i.e., $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma,$ for all $\alpha,\ \beta,\ \gamma \in \mathbf{C}$

(ii) *In* $\mathbf{C}$, *additive identity exists and is unique*

i.e., there exists unique $\tau \in \mathbf{C}$ *such that* $\alpha + \tau = \tau + \alpha = \alpha$ *for all* $\alpha \in \mathbf{C}$

This element τ *is called the additive identity.*

(iii) *Given* $\alpha \in \mathbf{C}$, *there exists a unique element* $\alpha' \in \mathbf{C}$ *such that* $\alpha + \alpha' = \alpha' + \alpha = \tau$.

The element α' *is called the additive inverse of* α .

(iv) *In* $\mathbf{C}$, *addition is commutative*

i.e., $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$

Proof : Let $\alpha, \beta, \gamma \in \mathbf{C}$. Let us take $\alpha = (a, b), \beta = (c, d)$ and $\gamma = (e, f)$. Then

$$
\begin{aligned}
\text{(i)}\quad \alpha + (\beta + \gamma) &= (a, b) + [(c, d) + (e, f)] \\
&= (a, b) + (c + e,\ d + f) \\
&= (a + c + e,\ b + d + f) \\
&= [(a + c) + e,\ (b + d) + f\,] \\
&= (a + c,\ b + d) + (e,\ f) \\
&= [(a, b) + (c, d)] + (e,\ f) \\
&= (\alpha + \beta) + \gamma
\end{aligned}
$$

$\therefore$ Addition is associative in $\mathbf{C}$.

(ii) *Let* $\tau = (0, 0)$. Then for all $\alpha = (a, b) \in \mathbf{C}$

$$\alpha + \tau = (a, b) + (0, 0) = (a + 0, b + 0)$$
$$= (a, b) = (0 + a, \ 0 + b) = (0, 0) + (a, b) = \tau + \alpha .$$

i.e., $\alpha + \tau = \alpha = \tau + \alpha$.

If $\theta \in \mathbf{C}$ is another element such that $\alpha + \theta = \theta + \alpha = \alpha$ for all $\alpha \in \mathbf{C}$,

then, by taking $\tau = \alpha$ we get $\theta = \theta + \tau = \tau$.

Hence in $\mathbf{C}$, additive identity is unique.

(iii) Let $\alpha \in \mathbf{C}$. Then $\alpha = (a, b)$ for some $a, b \in \mathbf{R}$.

Let $\alpha' = (-a, -b)$. Then

$$\alpha + \alpha' = (a, b) + (-a, -b) = (a - a, b - b) = (0, 0)$$
$$= (-a + a, \ -b + b) = \alpha' + \alpha$$

i.e., $\alpha + \alpha' = \tau = \alpha' + \alpha$

If $\beta \in \mathbf{C}$ be such that $\alpha + \beta = \tau = \beta + \alpha$, then

$$\beta = \beta + \tau = \beta + (\alpha + \alpha') = (\beta + \alpha) + \alpha' = \tau + \alpha' = \alpha'$$

Hence additive inverse is unique.

(iv) Let $\alpha = (a, b)$ and $\beta = (c, d)$. Then

$$\alpha + \beta = (a, b) + (c, d) = (a + c, \ b + d)$$
$$= (c + a, \ d + b) = (c, d) + (a, b) = \beta + \alpha.$$

$\therefore$ In $\mathbf{C}$, addition is commutative.

Note : The additive identity in $\mathbf{C}$ is denoted by 0. The additive inverse of $\alpha \in \mathbf{C}$ is denoted by $-\alpha$. We shall now establish the laws in Theorem 1.1.5 with multiplication as binary operation.

1.1.6 Theorem

(i) *In* $\mathbf{C}$, *multiplication is associative*: i.e., $\alpha . (\beta . \gamma) = (\alpha . \beta) . \gamma$ for all $\alpha, \beta, \gamma \in \mathbf{C}$

(ii) *In* $\mathbf{C}$, *there exists a unique element* η *satisfying* $\alpha . \eta = \eta . \alpha = \alpha$, for all $\alpha \in \mathbf{C}$.
This element η *is called the multiplicative identity.*

(iii) *In* $\mathbf{C}$, *multiplication is commutative*: i.e., $\alpha . \beta = \beta . \alpha$, for all $\alpha, \beta \in \mathbf{C}$

(iv) *In* $\mathbf{C}$, *distributive laws hold*: i.e., $\alpha . (\beta + \gamma) = \alpha . \beta + \alpha . \gamma$,

$$(\alpha + \beta) . \gamma = \alpha . \gamma + \beta . \gamma, \ \text{for all } \alpha, \beta, \gamma \in \mathbf{C}$$

Proof : Let us take $\alpha = (a, b)$, $\beta = (c, d)$, $\gamma = (e, f) \in \mathbf{C}$

Then (i) $\alpha \cdot (\beta \cdot \gamma) = (a, b) \cdot [(c, d) \cdot (e, f)]$

$$= (a, b) \cdot (ce - df, cf + de)$$

$$= [a(ce - df) - b(cf + de), a(cf + de) + b(ce - df)]$$

$$= [(ac - bd)e - (ad + bc)f, (ac - bd)f + (ad + bc)e]$$

$$= (ac - bd, ad + bc) \cdot (e, f)$$

$$= [(a, b) \cdot (c, d)] \cdot (e, f) = (\alpha \cdot \beta) \cdot \gamma.$$

(ii) Let $\eta = (1, 0)$. Then $\eta \in \mathbf{C}$. For any $\alpha \in \mathbf{C}$,

$$\eta \cdot \alpha = (1, 0) \cdot (a, b) = (a + 0, b + 0) = (a, b) = \alpha.$$

$$\alpha \cdot \eta = (a, b) \cdot (1, 0) = [a - 0, b + 0) = (a, b) = \alpha.$$

If $\xi \in \mathbf{C}$ is such that, $\alpha\xi = \xi\alpha = \alpha$ for all $\alpha \in \mathbf{C}$, then $\xi = \eta \cdot \xi = \eta$.

(iii) $\alpha . \beta = (a, b) \cdot (c, d) = (ac - bd, ad + bc) = (ca - db, da + cb)$,

(since $a, b, c, d \in \mathbf{R}$ and multiplication is commutative in $\mathbf{R}$)

$$= (c, d) \cdot (a, b) = \beta . \alpha.$$

(iv) $\alpha \cdot (\beta + \gamma) = (a, b) \cdot [(c, d) + (e, f)]$

$$= (a, b)(c + e, d + f)$$

$$= [a(c + e) - b(d + f), a(d + f) + b(c + e)]$$

$$= [ac + ae - bd - bf, ad + af + bc + be]$$

$$= (ac - bd, ad + bc) + (ae - bf, af + be)$$

$$= (a, b) \cdot (c, d) + (a, b) \cdot (e, f)$$

$$= \alpha \cdot \beta + \alpha \cdot \gamma$$

Note: From Theorems (1.1.5) and (1.1.6) it follows that the set of all complex numbers is a commutative ring with unity under the operations of addition and multiplication. This concept will be dealt with in higher classes. In (ii) the multiplicative identity η is denoted by 1.

As in the case of real number system, we can define the concept of division by non-zero complex numbers. We establish this through the following theorem:

1.1.7 Theorem (Division of complex numbers)

If $\alpha, \beta \in \mathbf{C}$ and $\beta \neq (0, 0)$, then there exists a unique $z \in \mathbf{C}$ such that $\alpha = \beta z$.

Proof: Let $\alpha = (a, b)$ and $\beta = (c, d)$. Since $\beta \neq (0, 0)$,

We have either $c \neq 0$ or $d \neq 0$ and hence $c^2 + d^2 \neq 0$.

$$\text{Let } z = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right). \text{ Then } z \in \mathbf{C}$$

$$\text{Now } \beta . z = (c, d) . \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right)$$

$$= \left(\frac{ac^2+bcd-bcd+ad^2}{c^2+d^2}, \frac{bc^2-adc+adc+bd^2}{c^2+d^2}\right)$$

$$= (a, b) = \alpha.$$

Let $(x, y) \in \mathbf{C}$ be such that $\alpha = \beta . (x, y)$

Then $a = cx - dy$ and $b = dx + cy$

Hence $x = \dfrac{ac+bd}{c^2+d^2}$ and $y = \dfrac{bc-ad}{c^2+d^2}$. Hence $(x, y) = z$.

Note : If $\beta \neq (0, 0)$, then the unique element $z \in \mathbf{C}$ such that $\beta z = \eta$, η being the multiplicative identity $(1, 0)$ in $\mathbf{C}$, is called the multiplicative inverse of β. It is denoted by β^{-1}. Note that, if $\beta = (a, b) \neq (0, 0)$, then $\beta^{-1} = \left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right)$.

1.1.8 Definition

For any $\alpha \in \mathbf{C}$, $\beta \in \mathbf{C}$, $\beta \neq (0, 0)$, the unique $z \in \mathbf{C}$ satisfying $\alpha = \beta z$ is denoted by α / β. From this we can define the division of $\alpha = (a, b)$ by $\beta = (c, d)$ as

$$z = \frac{\alpha}{\beta} = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right).$$

1.1.9 Example

If $\alpha = (2, 5)$, $\beta = (-1, 4)$ then $z = \dfrac{\alpha}{\beta} = \left(\dfrac{-2+20}{1+16}, \dfrac{-5-8}{1+16}\right) = \left(\dfrac{18}{17}, \dfrac{-13}{17}\right)$

1.1.10 Definition (Integral power of a complex number)

Let $z \in \mathbf{C}$. *We take* $z^1 = z$.

We define z^n *inductively for any* $z \neq 0$ *as follows:* $z^n = \begin{cases} z & \text{if } n = 1 \\ (1, 0) & \text{if } n = 0 \\ z^{n-1} \cdot z & \text{if } n > 0 \\ \left(z^{-1}\right)^{-n} & \text{if } n < 0 \end{cases}$

Exercise 1(a)

I. 1. If $z_1 = (2, -1), z_2 = (6, 3)$ find $z_1 - z_2$.

2. If $z_1 = (3, 5)$ and $z_2 = (2, 6)$ find $z_1 \cdot z_2$

3. Write the additive inverse of the following complex numbers:

(i) $\left(\sqrt{3}, 5\right)$ (ii) $(-6, 5) + (10, -4)$ (iii) $(2, 1)(-4, 6)$

II. 1. If $z_1 = (6, 3)$; $z_2 = (2, -1)$ find z_1 / z_2.

2. If $z = (\cos\theta, \sin\theta)$, find $\left(z - \dfrac{1}{z}\right)$

3. Write the multiplicative inverse of the following complex numbers:

(i) $(3, 4)$ (ii) $(\sin\theta, \cos\theta)$ (iii) $(7, 24)$ (iv) $(-2, 1)$

1.2 Representation of a complex number in the form $a + ib$

Let us recall that by a complex number we mean an ordered pair (a, b) of real numbers. For any real number a, $(a, 0)$ is a complex number. Also $(a, 0) + (b, 0) = (a + b, 0)$ and $(a, 0) \cdot (b, 0) = (ab, 0)$ for any $a, b \in \mathbf{R}$.

This situation helps us in identifying a real number a with the complex number $(a, 0)$. With this identification, addition and multiplication of real numbers are same as those of the corresponding complex numbers. For simplicity, we denote $(a, 0)$ by a and the complex number $(0, 1)$ by i.

Then, we find that $i \cdot i = (0, 1) \cdot (0, 1) = (-1, 0) = -1$.

Also, for any complex number (a, b), we have

$$(a, b) = (a, 0) + (0, b) = (a, 0) + [(b, 0) \cdot (0, 1)]$$
$$= a + bi$$

Under the above identification, the additive identity $O = (0, 0)$ in **C** coincides with the additive identity 0 in **R**. Similarly the multiplicative identity $1 = (1, 0)$ in **C** coincides with 1 in **R**.

Hence any complex number (a, b) can be represented as $a + bi$ or $a + ib$ where a and b are real numbers and i is a complex number such that $i^2 = -1$. (that is, i is a root of the equation $x^2 + 1 = 0$).

1.2.1 Real and imaginary parts of a complex number z

If $z = a + bi$; $a, b \in$ **R,** then a is called the real part of z and b, the imaginary part of z. These are denoted by Re (z) and Im (z) respectively. When Re $(z) = 0$, z is called purely imaginary.

Note that both the real and imaginary parts of z are real numbers. Also, with this representation of z in terms of real and imaginary parts, the addition and multiplication of complex numbers can be carried in the usual way, with the condition $i^2 = -1$. Thus

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$$

Observe that $a + bi = c + di$ if and only if $a = c$ and $b = d$ i.e., two complex numbers z and z' are equal if and only if Re (z) = Re (z') and Im (z) = Im (z').

1.2.2 Solved Problems

1. Problem: *Express* $\dfrac{4 + 2i}{1 - 2i} + \dfrac{3 + 4i}{2 + 3i}$ *in the form* $a + bi$, $a \in \mathbf{R}$, $b \in \mathbf{R}$.

Solution: $$\frac{4 + 2i}{1 - 2i} + \frac{3 + 4i}{2 + 3i} = \frac{(4 + 2i)(1 + 2i)}{(1 - 2i)(1 + 2i)} + \frac{(3 + 4i)(2 - 3i)}{(2 + 3i)(2 - 3i)}$$

$$= \frac{(4 - 4) + i(2 + 8)}{1^2 - (2i)^2} + \frac{(6 + 12) + i(8 - 9)}{2^2 - (3i)^2}$$

$$= \frac{10i}{5} + \frac{18 - i}{13} = \frac{130i + 90 - 5i}{65}$$

$$= \frac{90 + 125i}{65} = \frac{18}{13} + \frac{25}{13}i.$$

2. Problem: *Find the real and imaginary parts of the complex number* $\dfrac{a + ib}{a - ib}$.

Solution: $$\frac{a + ib}{a - ib} = \frac{(a + ib)^2}{(a - ib)(a + ib)} = \frac{a^2 - b^2 + 2iab}{a^2 + b^2}$$

$$= \frac{a^2 - b^2}{a^2 + b^2} + i\frac{2ab}{a^2 + b^2}$$

$\therefore$ Real part $= \dfrac{a^2 - b^2}{a^2 + b^2}$, imaginary part $= \dfrac{2ab}{a^2 + b^2}$.

3. Problem: *Express* $(1 - i)^3 (1 + i)$ *in the form of* $a + ib$.

Solution: $(1 - i)^3 (1 + i) = (1 - i)^2 (1 - i) (1 + i)$

$= (1 + i^2 - 2i)(1^2 - i^2)$

$= (1 - 1 - 2i)\ (1 + 1) = 2(0 - 2i)$

$= 0 - 4i = 0 + i(-4)$.

4. Problem: *Find the multiplicative inverse of* $7 + 24i$.

Solution: Since $(x + iy)\left(\dfrac{x - iy}{x^2 + y^2}\right) = 1$, it follows that the multiplicative inverse of $(x + iy)$ is $\dfrac{x - iy}{x^2 + y^2}$.

Hence the multiplicative inverse of $7 + 24i$ is

$$\frac{7 - 24i}{(7)^2 + (24)^2} = \frac{7 - 24i}{49 + 576} = \frac{7 - 24i}{625}.$$

5. Problem: *Determine the locus of* z, $z \neq 2i$, *such that* $\operatorname{Re}\left(\dfrac{z - 4}{z - 2i}\right) = 0$.

Solution: Let $z = x + iy$. Then

$$\frac{z - 4}{z - 2i} = \frac{x + iy - 4}{x + iy - 2i} = \frac{(x - 4) + iy}{x + i(y - 2)} = \frac{[(x - 4) + iy][x - i(y - 2)]}{[x + i(y - 2)][x - i(y - 2)]}$$

$$= \frac{(x^2 - 4x + y^2 - 2y) + i(2x + 4y - 8)}{x^2 + (y - 2)^2}.$$

Hence, real part of $\left(\dfrac{z - 4}{z - 2i}\right) = \dfrac{x^2 - 4x + y^2 - 2y}{x^2 + (y - 2)^2}$.

The ratio on the R.H.S. is zero i.e., $x^2 - 4x + y^2 - 2y = 0$ if and only if $(x - 2)^2 + (y - 1)^2 = 5$. Therefore

$$z \neq 2i \text{ and } \operatorname{Re}\left(\frac{z - 4}{z - 2i}\right) = 0 \Leftrightarrow (x, y) \neq (0, 2) \text{ and } (x - 2)^2 + (y - 1)^2 = 5.$$

Hence the locus of the given point representing the complex number is the circle with (2, 1) as centre and $\sqrt{5}$ units as radius except the point (0, 2).

6. Problem: *If* $4x + i(3x - y) = 3 - 6i$ *where* x *and* y *are real numbers, then find the values of* x *and* y.

Solution: We have $4x + i(3x - y) = 3 + i(-6)$. Equating the real and imaginary parts in the above equation, we get $4x = 3$, $3x - y = -6$. Upon solving the simultaneous equations, we get $x = 3/4$ and $y = 33/4$.

7. Problem: If $z = 2 - 3i$, *then show that* $z^2 - 4z + 13 = 0$.

Solution: $z = 2 - 3i \Rightarrow z - 2 = -3i \Rightarrow (z-2)^2 = (-3i)^2$

$$\Rightarrow z^2 + 4 - 4z = -9 \Rightarrow z^2 - 4z + 13 = 0.$$

1.2.3 Conjugate of a complex number

We have already defined a complex number and defined some fundamental operations on complex numbers. We have also represented a complex number z in the form $a + ib$ and called a, the real part and b, the imaginary part of z. In this section, we shall introduce the concepts of the conjugate of a complex number z, the square root and the n^{th} root of a complex number.

1.2.4 Definition

For any complex number $z = a + bi$, *we define the conjugate of* z *as* $a + (-b)i$ *and denote this by* $\bar{z}$.

To be more simple, we write $\bar{z} = a - bi$.

1.2.5 Note

(i) $\overline{a + ib} = a - ib$.

(ii) For $z \in \mathbf{C}$, $\left(\dfrac{z + \bar{z}}{2}\right)$ is the real part of z and $\left(\dfrac{z - \bar{z}}{2i}\right)$ is the imaginary part of z.

This means that if $z = a + ib$, then $a = \dfrac{z + \bar{z}}{2}$; $b = \dfrac{z - \bar{z}}{2i}$.

(iii) If $z = (a, 0)$, then $\bar{z} = \overline{(a, 0)} = \overline{(a + i0)} = (a, 0) = z$.

Hence z is a real number $\Leftrightarrow \bar{z} = z \Leftrightarrow Im(z) = 0$.

(iv) If $z = (0, b)$, then $\bar{z} = \overline{(0, b)} = (0 - ib) = -z$.

From this we have $\bar{z} = -z \Leftrightarrow Re(z) = 0$. In particular $\bar{i} = -i$.

1.2.6 Example

(i) If $z = 2+5i$, then $\bar{z} = \overline{2+5i} = 2-5i$.

(ii) If $z_1 = 15+8i$, $z_2 = 7-20i$ and $z = z_1 - z_2$, then

$z = (15+8i) - (7-20i) = (15-7)+i(8+20) = 8+28i$ and $\bar{z} = 8-28i$.

1.2.7 Theorem: *If $\alpha \in \mathbf{C}$, $\beta \in \mathbf{C}$, then the following results hold.*

(i) $\overline{\alpha+\beta} = \bar{\alpha}+\bar{\beta}$ (ii) $\overline{\alpha.\beta} = \bar{\alpha}.\bar{\beta}$

(iii) $\bar{\bar{\alpha}} = \alpha$ (iv) if $\beta \neq 0$, $\overline{(\alpha/\beta)} = \bar{\alpha}/\bar{\beta}$

Proof

(i) Let us take $\alpha, \beta \in \mathbf{C}$ as $\alpha = a+ib$, $\beta = c+id$, $a, b, c, d \in \mathbf{R}$.

Then $\alpha+\beta = (a+ib)+(c+id) = (a+c)+i(b+d)$

$$\overline{\alpha+\beta} = \overline{(a+c)+i(b+d)} = (a+c) - i(b+d)$$

$$= (a-ib)+(c-id) = \bar{\alpha}+\bar{\beta}.$$

(ii) $\overline{\alpha\beta} = \overline{(a+ib)(c+id)} = \overline{(ac-bd)+i(ad+bc)}$

$$= (ac-bd) - i(ad+bc)$$

$$= (a-ib)(c-id) = \bar{\alpha}.\bar{\beta}.$$

(iii) $\bar{\bar{\alpha}} = \left(\overline{\overline{a+ib}}\right) = \left(\overline{a-ib}\right) = a+ib = \alpha.$

(iv) Let $\alpha/\beta = \gamma$. Then $\alpha = \beta\gamma$, so that

$$\bar{\alpha} = \overline{\beta\gamma} = \bar{\beta}.\bar{\gamma}, \text{ (by (ii))}$$

$\therefore \bar{\gamma} = \bar{\alpha}/\bar{\beta}$, (since $\beta \neq 0$ and $\bar{\beta} \neq 0$). $\therefore \overline{(\alpha/\beta)} = \bar{\alpha}/\bar{\beta}$

1.2.8 Square roots of a complex number

Let $z \in \mathbf{C}$. If $w \in \mathbf{C}$ is such that $w^2 = z$, then w is called a square root of z. We now find square root of $z = a+ib$,

Case (i) : Suppose $b = 0$. Then, if $a > 0$, $w^2 = z = a$ or $w = \pm\sqrt{a}$,

if $a < 0$, $w^2 = z = a$ or $w = \pm i\sqrt{-a}$

Case (ii) : Suppose $b \neq 0$.

Let $(x + iy)$ be such that $(x + iy)^2 = a + ib$

Then $a + i\,b = x^2 - y^2 + 2\,i\,x\,y.$

On equating the real and imaginary parts on both sides of the above equation, we obtain $a = x^2 - y^2$, $b = 2xy$.

$$\therefore\ (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = a^2 + b^2,$$

and $x^2 + y^2 = \sqrt{a^2 + b^2}$. Since $x^2 - y^2 = a,$ we have,

$$2x^2 = a + \sqrt{a^2 + b^2} \quad \text{and} \quad 2y^2 = \sqrt{a^2 + b^2} - a$$

$$\text{Hence}\quad x = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}},\ y = \pm\sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}}$$

Since $2xy = b$, both x and y have the same sign – either both positive or both negative if $b > 0$ and x and y have opposite signs if $b < 0$.

$$\text{Hence}\ x + iy = \begin{cases} \pm\left[\sqrt{\dfrac{\sqrt{a^2 + b^2} + a}{2}} + i\sqrt{\dfrac{\sqrt{a^2 + b^2} - a}{2}}\right], & \text{if}\ \ b > 0 \\[2ex] \pm\left[\sqrt{\dfrac{\sqrt{a^2 + b^2} + a}{2}} - i\sqrt{\dfrac{\sqrt{a^2 + b^2} - a}{2}}\right], & \text{if}\ \ b < 0 \end{cases}$$

If $x + iy$ is defined as above, then it can be verified that $(x + iy)^2 = a + ib$, so that $x + iy$ is a square root of $a + ib$.

1.2.9 Note

(i) For a positive real number a, the notation $\sqrt{a}$ is used to denote the square root of a. For a complex number z, the notation $z^{1/2}$ stands for any one of the square roots of z. But it is also customary to write $\sqrt{z}$ for $z^{1/2}$.

(ii) If $\sqrt{a + ib} = \pm(x + iy)$, then $\sqrt{a - ib} = \pm(x - iy)$.

(iii) $\sqrt{i} = \pm\dfrac{1+i}{\sqrt{2}}$, $\sqrt{-i} = \pm\dfrac{1-i}{\sqrt{2}}$.

(iv) $\sqrt{a+ib} + \sqrt{a-ib} = \pm\sqrt{2a + 2\sqrt{a^2+b^2}}$ or $\pm i\sqrt{2\sqrt{a^2+b^2} - 2a}$.

1.2.10(a) Definition (n^{th} root of a complex number)

Let n be a positive integer and $z \in \mathbf{C}$. If $w \in \mathbf{C}$ is such that $w^n = z$, then w is called an n^{th} root of z and is denoted by $z^{1/n}$.

In lesson 2, we shall prove that a non-zero complex number has exactly n distinct n^{th} roots.

(b) Definition

Let z be a non–zero complex number, n a positive integer and m an integer. Then $z^{m/n}$ is defined as $\left(z^m\right)^{1/n}$.

1.2.11 Note: If $z \neq 0$, $(z^m)^{1/n}$ has n distinct values

The following notation will be helpful in solving some problems connected with locus later.

Circle : If the centre is (h, k) and radius r, then the equation of the circle is

$(x - h)^2 + (y - k)^2 = r^2$.

The standard form of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, $(g^2 + f^2 - c > 0)$

Here the centre is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.

Hyperbola : The general form of a hyperbola with centre (h, k) is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$. The standard form of hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, if the centre is $(0, 0)$.

We will learn in detail about circles, and conics in coordinate geometry.

1.2.12 Solved Problems

1. Problem: *Find the complex conjugate of* $(3 + 4i)\ (2 - 3i)$.

Solution: The given complex number $(3 + 4i)\ (2 - 3i) = 6 - 9i + 8i + 12 = 18 - i$. Its complex conjugate $= 18 + i$.

2. Problem: *Show that* $z_1 = \dfrac{2 + 11i}{25}$, $z_2 = \dfrac{-2 + i}{(1-2i)^2}$ *are conjugate to each other.*

Solution: $\dfrac{-2+i}{(1-2i)^2} = \dfrac{-2+i}{1+4i^2-4i} = \dfrac{-2+i}{-(3+4i)}$

$$= \frac{2-i}{3+4i} = \frac{(2-i)(3-4i)}{(3+4i)(3-4i)} = \frac{2-11i}{25}.$$

Since this complex number is the conjugate of $\frac{2+11i}{25}$, the given complex numbers z_1, z_2 are conjugate to each other.

3. Problem: *Find the square roots of* $(-5+12i)$.

Solution: From 1.2.8, we have

$$\sqrt{a+ib} = \pm\left[\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right]$$

Here $a = -5$ and $b = 12$. Hence

$$\sqrt{-5+12i} = \pm\left[\sqrt{\frac{\sqrt{25+144}+(-5)}{2}} + i\sqrt{\frac{\sqrt{25+144}-(-5)}{2}}\right]$$

$$= \pm\left[\sqrt{\frac{13-5}{2}} + i\sqrt{\frac{13+5}{2}}\right] = \pm(2+i.3).$$

We write this as $\pm(2+3i)$.

Exercise 1(b)

I. 1. Write the following complex numbers in the form $A + iB$.

(i) $(2-3i)(3+4i)$ (ii) $(1+2i)^3$ (iii) $\frac{a-ib}{a+ib}$

(iv) $\frac{4+3i}{(2+3i)(4-3i)}$ (v) $(-\sqrt{3}+\sqrt{-2})(2\sqrt{3}-i)$ (vi) $(-5i)\left(\frac{i}{8}\right)$

(vii) $(-i)\,2i$ (viii) i^9 (ix) i^{-19}

(x) $3(7+7i)+i(7+7i)$ (xi) $\frac{2+5i}{3-2i}+\frac{2-5i}{3+2i}$

2. Write the conjugate of the following complex numbers.

(i) $(3+4i)$ (ii) $(15+3i)-(4-20i)$ (iii) $(2+5i)(-4+6i)$

(iv) $\frac{5i}{7+i}$

3. Simplify

(i) $i^2+i^4+i^6+ \ldots + (2n+1)$ terms (ii) $i^{18}-3.i^7+i^2(1+i^4)(-i)^{26}$

4. Find a square root for the following complex numbers.

(i) $7+24i$ (ii) $-8-6i$ (iii) $3+4i$

(iv) $-47+i.8\sqrt{3}$

5. Find the multiplicative inverse of the following complex numbers.

(i) $\sqrt{5}+3i$ (ii) $-i$ (iii) i^{-35}

II. 1. (i) If $(a+ib)^2=x+iy$, find x^2+y^2.

(ii) If $x+iy=\dfrac{3}{2+\cos\theta+i\sin\theta}$ then, show that $x^2+y^2=4x-3$

(iii) If $x+iy=\dfrac{1}{1+\cos\theta+i\sin\theta}$ then, show that $4x^2-1=0$.

(iv) If $u+iv=\dfrac{2+i}{z+3}$ and $z=x+iy$ then find u, v.

2. (i) If $z=3-5i$, then show that $z^3-10z^2+58z-136=0$.

(ii) If $z=2-i\sqrt{7}$, then show that $3z^3-4z^2+z+88=0$.

(iii) Show that $\dfrac{2-i}{(1-2i)^2}$ and $\dfrac{-2-11i}{25}$ are conjugate to each other.

3. (i) If $(x-iy)^{1/3}=a-ib$, then show that $\dfrac{x}{a}+\dfrac{y}{b}=4(a^2-b^2)$.

(ii) Write $\left(\dfrac{a+ib}{a-ib}\right)^2-\left(\dfrac{a-ib}{a+ib}\right)^2$ in the form $x+iy$.

(iii) If x and y are real numbers such that $\dfrac{(1+i)x-2i}{3+i}+\dfrac{(2-3i)y+i}{3-i}=i$, then determine the values of x and y.

4. (i) Find the least positive integer n, satisfying $\left(\dfrac{1+i}{1-i}\right)^n=1$.

(ii) If $\left(\dfrac{1+i}{1-i}\right)^3-\left(\dfrac{1-i}{1+i}\right)^3=x+iy$, find x and y.

(iii) Find the real values of θ in order that $\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$ is *a*

(a) real number (ii) purely imaginary number

(iv) Find the real values of x and y if $\dfrac{x-1}{3+i}+\dfrac{y-1}{3-i}=i$.

1.3 Modulus and Amplitude of a complex number - Illustrations

In this section, we shall first define the modulus of a complex number. Then we shall introduce the concept of amplitude of a complex number and its principal amplitude. We shall learn how to express a given complex number in the modulus – amplitude or polar form.

1.3.1 Definition (Modulus or Absolute value of a complex number)

The modulus or absolute value of a complex number $z = x + iy$ *is defined to be the non–negative real number* $\sqrt{x^2 + y^2}$. *It is denoted by* $|z|$ *and is called mod z. Geometrically the modulus* $|z| = |x + iy|$ *is the distance from the origin (0, 0) to the point* (x, y).

1.3.2 Note

(i) $|z| \geq 0 \ \forall \ z \in \mathbf{C}$

(ii) Let $z = (x, y) \in \mathbf{C}$. Then $|z| = 0 \iff \sqrt{x^2 + y^2} = 0$

$$\iff x^2 + y^2 = 0 \iff x = y = 0 \iff z = 0.$$

1.3.3 Example : Let $z = 3 + i$. Then $|z| = |3 + i| = \sqrt{3^2 + 1^2} = \sqrt{10}$.

1.3.4 Theorem : *If* $\alpha \in \mathbf{C}$, $\beta \in \mathbf{C}$ *then*

(i) $|\alpha| = |\bar{\alpha}|$ (ii) $|\alpha|^2 = \alpha\bar{\alpha}$ (iii) $|\alpha\beta| = |\alpha|\,|\beta|$

Proof : Let $\alpha = x + iy;\ x, y \in \mathbf{R}$. Then $\bar{\alpha} = x - iy = x + i(-y)$

(i) $|\bar{\alpha}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |\alpha|$

(ii) $\alpha\bar{\alpha} = (x + iy)(x - iy) = x^2 + y^2 = |\alpha|^2$

(iii) $|\alpha\beta|^2 = (\alpha\beta)\,(\overline{\alpha\beta})$, (from above result (ii))

$= \alpha\beta\,.\,\bar{\alpha}\bar{\beta}$, (from Theorem 1.2.7 (ii))

$= \alpha\bar{\alpha}\,.\,\beta\bar{\beta} = |\alpha|^2\,|\beta|^2$, (from above result (ii))

$\therefore |\alpha\beta| = |\alpha|\,|\beta|$.

1.3.5 Theorem: *Let* $\alpha \in \mathbf{C},\ \beta \in \mathbf{C}$. *Then*

(i) $|Re(\alpha)| \le |\alpha|;\quad |\mathrm{Im}(\alpha)| \le |\alpha|$

(ii) $|\alpha + \beta|^2 = |\alpha|^2 + |\beta|^2 + 2\,Re(\alpha\bar{\beta})$

(iii) $|\alpha + \beta|^2 + |\alpha - \beta|^2 = 2\left(|\alpha|^2 + |\beta|^2\right)$

(iv) $|\alpha + \beta| \le |\alpha| + |\beta|$

Proof: Let $\alpha = a + ib$; $\beta = c + id$

(i) Since $|\alpha^2| = a^2 + b^2$, $a^2 \le |\alpha|^2$

Hence $|Re(\alpha)| = |a| \le |\alpha|$

In a similar way, we can show that $|\mathrm{Im}(\alpha)| \le |\alpha|$

(ii) $|\alpha + \beta|^2 = (\alpha + \beta)\left(\overline{\alpha + \beta}\right)$; from Theorem 1.3.4 (ii)

$= (\alpha + \beta)\left(\bar{\alpha} + \bar{\beta}\right)$; from Theorem 1.2.7 (i)

$= \alpha\bar{\alpha} + \alpha\bar{\beta} + \beta\bar{\alpha} + \beta\bar{\beta}.$

$= \alpha\bar{\alpha} + \beta\bar{\beta} + \alpha\bar{\beta} + \left(\overline{\alpha\bar{\beta}}\right)$, (from Theorem 1.2.7 (ii), (iii))

$= |\alpha|^2 + |\beta|^2 + 2\,Re\left(\alpha\bar{\beta}\right).$

(iii) $|\alpha + \beta|^2 + |\alpha - \beta|^2 = |\alpha|^2 + |\beta|^2 + 2\,Re\left(\alpha\bar{\beta}\right) + |\alpha|^2 + |\beta|^2 + 2\ Re\left(\alpha\left(-\bar{\beta}\right)\right)$,

(from result (ii))

$= |\alpha|^2 + |\beta|^2 + 2\ Re\left(\alpha\,\bar{\beta}\right) + |\alpha|^2 + |\beta|^2 - 2\ Re\left(\alpha\,\bar{\beta}\right)$

$= 2\left(|\alpha|^2 + |\beta|^2\right).$

(iv) We have $|\alpha + \beta|^2 = |\alpha|^2 + |\beta|^2 + 2\,Re\left(\alpha\bar{\beta}\right)$, from (ii)

$\le |\alpha|^2 + |\beta|^2 + 2\left|\alpha\bar{\beta}\right|$

$= |\alpha|^2 + |\beta|^2 + 2|\alpha|\,|\beta|$, (from (i) and (iii) of Theorem 1.3.4)

$= \left(|\alpha| + |\beta|\right)^2$

Hence $|\alpha + \beta| \le |\alpha| + |\beta|$.

1.3.6 Cartesian coordinates and Polar coordinates

Let us consider a plane. Let O be a fixed point and OX be a fixed ray in the plane. The point O is called the initial point and the ray OX is called the initial line. Let P be a point in the plane $P \neq 0$. Let $OP = r$. Let θ denote the angle which OP makes with the initial line measured in the anticlockwise sense. We note that, depending on the position of the point P in the plane, r can vary from 0 to ∞ (excluding ∞) and θ can vary from 0 to 2π (excluding 2π). The ordered pair (r, θ) is called the polar representation of the point P and r, θ are called the polar coordinates of P.

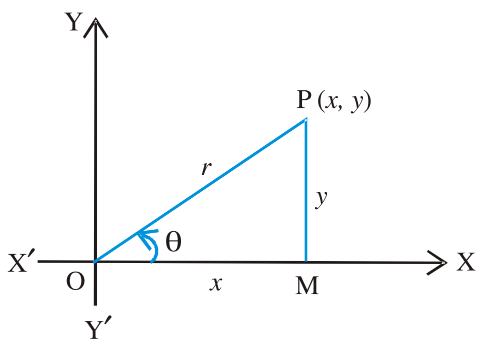


Fig. 1.1(a)

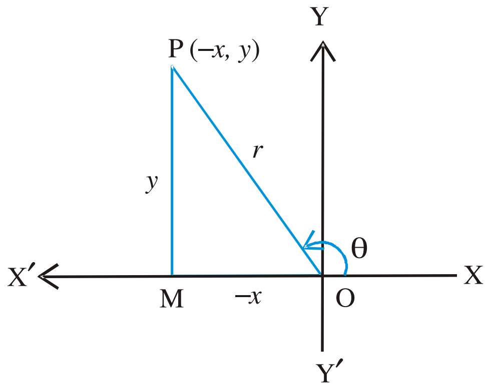


Fig. 1.1(b)

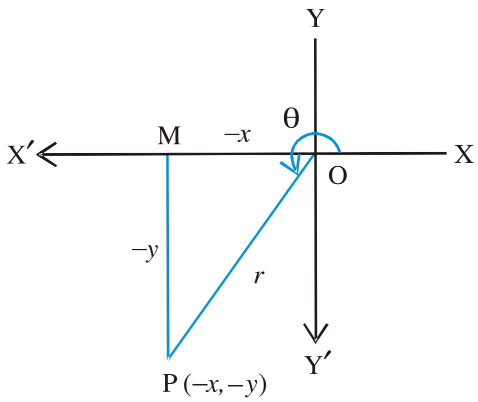


Fig. 1.1(c)

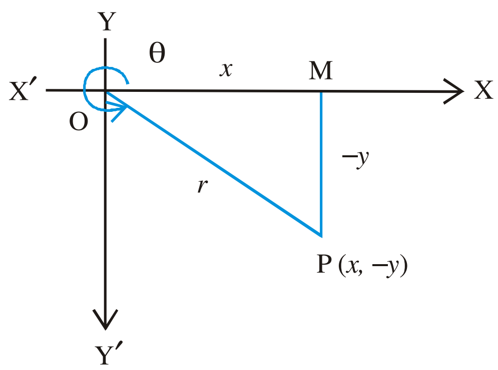


Fig. 1.1(d)

Since P is different from O, θ is uniquely determined.

We shall now choose OY in the plane so that OY makes an angle $\frac{\pi}{2}$ with OX, measured in the anti-clockwise sense. Let (x, y) be the cartesian coordinates of P with respect to the frame work OXY. Let M be the foot of the perpendicular from P to the X–axis.

Then $x = +(\mathrm{OM})$ or $-(\mathrm{OM})$ according as M lies on positive or negative side of the X–axis and $y = +(\mathrm{MP})$ or $-(\mathrm{MP})$ according as P lies above or below the X–axis.

Let (r, θ) be the polar representation of P. From Δ POM we see that

$\mathrm{OM} = (\mathrm{OP})\cos\angle\mathrm{POM}$ and $\mathrm{PM} = (\mathrm{OP})\sin\angle\mathrm{POM}$. We note that $\angle\mathrm{POM} = \theta$ or $(\pi - \theta)$ or $(\theta - \pi)$ or $(2\pi - \theta)$ according as point P lies in the first or the second or the third or the fourth quadrants.

When P lies in the first or fourth quadrant, $\cos\angle\mathrm{POM} = \cos\theta$ and M lies on the positive side of the X–axis, so that $x = \mathrm{OM}$ (see figures 1.1(a) and 1.1(d)). When P lies in the second or third quadrants, $\cos\angle\mathrm{POM} = \cos\theta$ and M lies on the negative side of the X–axis so that $x = -(\mathrm{OM})$ (see figures 1.1(b) and 1.1(c)). Hence $x = r\cos\theta$ in all cases. When P lies in the first two quadrants, $\sin\angle\mathrm{POM} = \sin\theta$ and $y = \mathrm{PM}$.

When P lies in the third or fourth quadrants, then $\sin\angle\mathrm{POM} = -\sin\theta$ and $y = -(\mathrm{MP})$. Hence $y = r\sin\theta$ in all the cases.

Thus we have $x = r\cos\theta$ and $y = r\sin\theta$ ... (1)

In the above discussion we tacitly assumed that P is different from the origin O. When P coincides with O, we have $x = 0$, $y = 0$ and $r = 0$ and θ can be any number in $[0, 2\pi)$ and equations (1) remain valid in this case also.

We know that the mapping which associates to each point in the plane, its cartesian coordinates (relative to the frame work of OXY) is in one to one correspondence between the set of all points in the plane and $\mathbf{R}\times\mathbf{R}$. Since $\mathbf{C} = \mathbf{R}\times\mathbf{R}$, to any complex number $z = x + iy$, there corresponds a unique point in the plane with Cartesian coordinates (x, y) and vice-versa. We note that a complex number z lies on the X–axis if and only if z is a real number and that it lies on the Y–axis if and only if it is purely imaginary. Hence the X–axis is often referred to as the real axis and the Y–axis as the imaginary axis.

If $z = (x, y) \in \mathbf{C}$ and P is the point in the plane with cartesian coordinates (x, y) and if (r, θ) are the polar coordinates of P, then we have

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

further $|z| = \sqrt{x^2 + y^2} = \sqrt{r^2\left(\cos^2\theta + \sin^2\theta\right)} = \sqrt{r^2} = r$, (since $r \geq 0$).

1.3.7 Definition (Amplitude of a complex number, principal amplitude)

In view of equations (1) of 1.3.6, it follows that for any given non–zero complex number $z = (x, y)$ there exists a $\theta \in \mathbf{R}$ such that

$$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}; \quad \sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}. \qquad \text{... (2)}$$

Any real number θ satisfying the pair of equations (2), is called an amplitude or argument of z. However, for a given complex number $z \neq 0$ there exists a unique θ in the interval $[-\pi, \pi)$ satisfying equation (2). We call such θ the principal amplitude or principal argument of z and denote it by Arg z. For the principal amplitude θ, $\tan\theta = y/x$ and therefore $\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right)$, if $x \neq 0$.

If n is an integer and θ is an amplitude of z, then $(2n\pi + \theta)$ is also an amplitude of z and is called the general value of the amplitude of z.

1.3.8 Note

The principal amplitude of $z = (x, y)$ lies in $(0, \pi/2)$ or $(\pi/2, \pi)$ or $(-\pi, -\pi/2)$ or $(-\pi/2, 0)$ according as the point (x, y) lies in the first or the second or the third or the fourth quadrant and is not on the axes.

1.3.9 Definition (Modulus - Amplitude form)

$r(\cos\theta + i\sin\theta)$, $\theta \in (-\pi, \pi]$, *is called the polar form or modulus amplitude form of the complex number* $z = (x, y)$.

1.3.10 Example

Find the modulus and the principal amplitude of the complex number $\left(-1-\sqrt{3}\,i\right)$.

Solution : Let $-1-\sqrt{3}\,i = r(\cos\theta + i\sin\theta)$

Then $r\cos\theta = -1$ and $r\sin\theta = -\sqrt{3}$

$$\therefore r = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{1+3} = 2.$$

Hence $\cos\theta = -\dfrac{1}{2}$, $\sin\theta = \dfrac{-\sqrt{3}}{2}$.

Since the point $\left(-1-\sqrt{3}\,i\right)$ lies in the third quadrant, we look for a solution of the above equation that lies in $[-\pi, -\pi/2)$. We find that $\theta = -\pi + \pi/3 = -2\pi/3$ is such a solution.

$\therefore$ For the given complex number, modulus $= 2$ and principal amplitude $= -2\pi/3$.

1.3.11 Note

(i) We write $(\cos\theta + i\sin\theta)$ in the simplified form as cis θ. You will learn that cis $\theta = e^{i\theta}$ later. Since $i^2 = -1$ and $\sin^2\theta + \cos^2\theta = 1$, we have

$$\cos^2\theta - i^2\sin^2\theta = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = 1.$$

From this $\text{cis}\,\theta = \dfrac{1}{\cos\theta - i\sin\theta}$.

c is $(-\theta) = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta = \dfrac{1}{\cos\theta + i\sin\theta} = \dfrac{1}{\text{cis}\,\theta}$.

(ii) We note that $\sin\theta + i\cos\theta = \cos\left(\dfrac{\pi}{2} - \theta\right) + i\sin\left(\dfrac{\pi}{2} - \theta\right) = \text{cis}\left(\dfrac{\pi}{2} - \theta\right)$.

1.3.12 Some operations on complex numbers in the modulus – amplitude form

Let $z_1 = (x, y) = r_1(\cos\theta_1 + i\sin\theta_1)$; $z_2 = (p, q) = r_2(\cos\theta_2 + i\sin\theta_2)$.

We shall now express $z_1 z_2$; z_1/z_2, $z_2 \neq 0$ in the modulus – amplitude form:

$$z_1 = (x, y) = r_1(\cos\theta_1 + i\sin\theta_1) \Rightarrow x = r_1\cos\theta_1,\ y = r_1\sin\theta_1.$$

$$z_2 = (p, q) = r_2(\cos\theta_2 + i\sin\theta_2) \Rightarrow p = r_2\cos\theta_2,\ q = r_2\sin\theta_2.$$

$$\begin{aligned} z_1 z_2 = (x, y)\ (p, q) &= (xp - yq,\ \ xq + yp) \\ &= [\,(r_1\cos\theta_1 \,.\, r_2\cos\theta_2 - r_1\sin\theta_1 \,.\, r_2\sin\theta_2), \\ &\qquad (r_1\cos\theta_1 \,.\, r_2\sin\theta_2 + r_1\sin\theta_1 \,.\, r_2\cos\theta_2)] \\ &= [r_1 r_2\cos(\theta_1 + \theta_2),\ r_1 r_2\sin(\theta_1 + \theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2), \sin(\theta_1 + \theta_2)] \\ &= r_1 r_2\, cis\,(\theta_1 + \theta_2). \end{aligned}$$

In a similar way, we can prove by mathematical induction that, when n is a positive integer,

$$z_1 \,.\, z_2 \ldots z_n = r_1 r_2 \ldots r_n \text{ cis } (\theta_1 + \theta_2 + \ldots + \theta_n),$$

$$\frac{z_1}{z_2} = \frac{x + iy}{p + iq} = \frac{(x, y)}{(p, q)} = \left(\frac{xp + yq}{p^2 + q^2}, \frac{yp - xq}{p^2 + q^2}\right)$$

$$= \left(\frac{r_1\cos\theta_1 \,.\, r_2\cos\theta_2 + r_1\sin\theta_1 \,.\, r_2\sin\theta_2}{r_2^2\cos^2\theta_2 + r_2^2\sin^2\theta_2}, \frac{r_1\sin\theta_1 \,.\, r_2\cos\theta_2 - r_1\cos\theta_1 \,.\, r_2\sin\theta_2}{r_2^2\cos^2\theta_2 + r_2^2\sin^2\theta_2}\right)$$

$$= \left(\frac{r_1 r_2\cos(\theta_1 - \theta_2)}{r_2^2}, \frac{r_1 r_2\sin(\theta_1 - \theta_2)}{r_2^2}\right)$$

$$= r_1/r_2\left[\cos(\theta_1 - \theta_2), \sin(\theta_1 - \theta_2)\right] = \frac{r_1}{r_2} cis\,(\theta_1 - \theta_2).$$

1.3.13 Note

(i) $cis\,\theta_1 \,.\, cis\,\theta_2 = cis\,(\theta_1 + \theta_2)$.

(ii) $\dfrac{\text{cis}\,\theta_1}{\text{cis}\,\theta_2} = \text{cis}\left(\theta_1 - \theta_2\right)$.

(iii) $mod\,(z_1 \,.\, z_2) = (mod\ z_1)\,.\,(mod\ z_2)$.

(iv) $Arg\,(z_1 \,.\, z_2) = Arg\,z_1 + Arg\,z_2 + n\pi$, for some $n \in \{-1, 0, 1\}$.

(v) $Arg\,(z_1 / z_2) = Arg\,z_1 - Arg\,z_2 + n\,\pi$, for some $n \in \{-1, 0, 1\}$.

1.3.14 Solved Problems

1. Problem: *Write* $z = -\sqrt{7} + i\sqrt{21}$ *in the polar form.*

Solution: If $z = -\sqrt{7} + i\sqrt{21} = x + i\,y$,

then $x = -\sqrt{7}$ and $y = \sqrt{21}$, $r = \sqrt{x^2 + y^2} = \sqrt{7 + 21} = \sqrt{28} = 2\sqrt{7}$

$\tan\theta = y/x = \sqrt{21}/-\sqrt{7} = -\sqrt{3}$.

Since the given point lies in the second quadrant, we look for a solution of $\tan\theta = -\sqrt{3}$ which lies in $\left[\dfrac{\pi}{2}, \pi\right]$. We find that $\theta = \dfrac{2\pi}{3}$ is such a solution.

$$\therefore -\sqrt{7} + i\sqrt{21} = 2\sqrt{7}\,cis\,\frac{2\pi}{3} \quad \text{or} \quad 2\sqrt{7}\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right].$$

2. Problem: *Express* $-1 - i$ *in polar form with principal value of the amplitude.*

Solution: Let $-1 - i = r\,(\cos\theta + i\sin\theta)$.

Then $-1 = r\cos\theta$, $-1 = r\sin\theta$ and $\tan\theta = 1$ ... (1)

$$\therefore r^2 = 2, \quad \text{i.e., } r = \pm\sqrt{2}. \quad \text{Since } r \text{ is positive, } r = \sqrt{2}.$$

Since 'θ' satisfies $-\pi \le \theta < \pi$, the value of θ satisfying the equation (1) is $\theta = -3\pi/4$.

$$\therefore -1 - i = \sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right].$$

3. Problem: *If the amplitude of* $\left(\dfrac{z-2}{z-6i}\right)=\dfrac{\pi}{2}$, *find its locus.*

Solution : Let $z = x+iy$. Then $\dfrac{z-2}{z-6i} = \dfrac{x+iy-2}{x+iy-6i}$

$$= \frac{(x-2)+iy}{x+i(y-6)} = \frac{\left[(x-2)+iy\right]\left[x-i(y-6)\right]}{\left[x+i(y-6)\right]\left[x-i(y-6)\right]}$$

$$= \frac{x(x-2)+y(y-6)}{x^2+(y-6)^2} + i\,\frac{xy-(x-2)(y-6)}{x^2+(y-6)^2} = a+ib\ (say)$$

Then $a = \dfrac{x(x-2)+y(y-6)}{x^2+(y-6)^2}$, $b = \dfrac{xy-(x-2)(y-6)}{x^2+(y-6)^2}$

But by the hypothesis, amplitude of $a+ib=\pi/2$. Hence $a=0$ and $b\geq 0$

$$\therefore\ x(x-2) + y(y-6)=0 \quad \text{or} \quad x^2+y^2-2x-6y=0 \qquad \text{... (1)}$$

$$\text{and } 3x+y-6 \geq 0. \qquad \text{... (2)}$$

The points satisfying (1) and (2) constitute the arc of the circle $x^2+y^2-2x-6y=0$ intercepted by the diameter $3x+y-6=0$ not containing the origin and excluding the points (0, 6) and (2, 0). Hence this arc is the required locus.

4. Problem: *Show that the equation of any circle in the complex plane is of the form* $z\bar{z}+b\bar{z}+\bar{b}z+c=0,\ (b\in \mathbf{C};\ c\in \mathbf{R})$.

Solution: Assume the general form of the equation of a circle in Cartesian coordinates as

$$x^2+y^2+2gx+2fy+c=0,\ (g, f\in \mathbf{R}). \qquad \text{... (1)}$$

To write this equation in the complex variable form,

let $(x, y) = z$. Then $\dfrac{z+\bar{z}}{2}=x$, $\dfrac{z-\bar{z}}{2i}=y=-\dfrac{i(z-\bar{z})}{2}$,

$$x^2+y^2=|z|^2=z\bar{z}.$$

Substituting these results in equation (1), we obtain

$$z\bar{z}+g(z+\bar{z})+f(z-\bar{z})(-i)+c=0$$

$$\text{i.e., } z\bar{z}+(g-if)z+(g+if)\bar{z}+c=0. \qquad \text{... (2)}$$

If $g+if=b$, then equation (2) can be written as $z\bar{z}+\bar{b}z+b\bar{z}+c=0$.

5. Problem: *Show that the complex numbers z satisfying $z^2 + \bar{z}^2 = 2$ constitute a hyperbola.*

Solution : Substituting $z = x + iy$ in the given equation $z^2 + \bar{z}^2 = 2$, we obtain the Cartesian form of the given equation.

$$\therefore (x + iy)^2 + (x - iy)^2 = 2 \text{ i.e., } x^2 - y^2 + 2ixy + x^2 - y^2 - 2ixy = 2$$

$$\text{or } 2x^2 + 2(iy)^2 = 2 \quad \text{i.e., } x^2 - y^2 = 1.$$

Since this equation denotes a hyperbola, all the complex numbers satisfying $z^2 + \bar{z}^2 = 2$ constitute the hyperbola $x^2 - y^2 = 1$.

Exercise 1(c)

I. 1. Express the following complex numbers in modulus - amplitude form

(i) $1 - i$ (ii) $1 + i\sqrt{3}$ (iii) $-\sqrt{3} + i$

(iv) $-1 - i\sqrt{3}$

2. Simplify $-2i(3+i)(2+4i)(1+i)$ and obtain the modulus of that complex number.

3. (i) If $z \neq 0$, find $\operatorname{Arg} z + \operatorname{Arg} \bar{z}$.

(ii) If $z_1 = -1$ and $z_2 = -i$, then find $\operatorname{Arg}(z_1 z_2)$

(iii) If $z_1 = -1$ and $z_2 = i$, then find $\operatorname{Arg}\left(\frac{z_1}{z_2}\right)$.

4. (i) If $(\cos 2\alpha + i \sin 2\alpha)(\cos 2\beta + i \sin 2\beta) = \cos\theta + i \sin\theta$, then find the value of θ.

(ii) If $\sqrt{3} + i = r(\cos\theta + i\sin\theta)$, then find the value of θ in radian measure.

(iii) If $x + iy = \operatorname{cis}\alpha . \operatorname{cis}\beta$, then find the value of $x^2 + y^2$.

(iv) If $\frac{z_2}{z_1}, z_1 \neq 0$, is an imaginary number then find the value of $\left|\frac{2z_1 + z_2}{2z_1 - z_2}\right|$.

(v) If $(\sqrt{3} + i)^{100} = 2^{99}(a + ib)$, then show that $a^2 + b^2 = 4$.

5. (i) If $z = x + iy$ and $|z| = 1$, then find the locus of z.

(ii) If the amplitude of $(z - 1)$ is $\frac{\pi}{2}$, then find the locus of z.

(iii) If the Arg $\bar{z}_1$ and Arg z_2 are $\frac{\pi}{5}$ and $\frac{\pi}{3}$ respectively, then find $(\text{Arg}\, z_1 + \text{Arg}\, z_2)$.

(iv) If $z = \frac{1+2i}{1-(1-i)^2}$, then find Arg$(z)$.

II. 1. Simplify the following complex numbers and find their modulus.

(i) $\frac{(2+4i)(-1+2i)}{(-1-i)(3-i)}$ (ii) $\frac{(1+i)^3}{(2+i)(1+2i)}$

2. (i) If $(1-i)(2-i)(3-i) \ldots (1-ni) = x - iy$, then prove that $2.5 \,.\, 10 \ldots (1+n^2) = x^2 + y^2$.

(ii) If the real part of $\frac{z+1}{z+i}$ is 1, then find the locus of z.

(iii) If $|z-3+i| = 4$, determine the locus of z.

(iv) If $|z + ai| = |z - ai|$, then find the locus of z.

3. If $z = (x+iy)$ and if the point P in the Argand plane represents z, then describe geometrically the locus of P satisfying the equations

(i) $|2z-3| = 7$ (ii) $|z|^2 = 4\,\text{Re}(z+2)$

(ii) $|z+i|^2 - |z-i|^2 = 2$ (iv) $|z+4i| + |z-4i| = 10$

4. (i) If z_1, z_2 are two nonzero complex numbers satisfying $|z_1 + z_2| = |z_1| + |z_2|$, then show that Arg z_1 − Arg $z_2 = 0$.

(ii) If $z = x + iy$ and the point P represents z in the Argand plane and $\left|\frac{z-a}{z+a}\right| = 1, \text{Re}(a) \neq 0$, then find the locus of P.

1.4 Geometrical and Polar representation of a complex number in Argand plane - Argand diagram

In the earlier discussion a complex number is defined as an ordered pair of real numbers and also defined the fundamental operations of addition, subtraction, multiplication and division algebraically.

In this section we give a geometrical representation to the complex number and use this representation to find the sum, difference, product and quotient of two complex numbers.

1.4.1 Geometrical representation of complex numbers

Gauss was one of the mathematicians who first thought that complex numbers can be represented on a two-dimensional plane. Gauss introduced a pair of perpendicular coordinate axes and fixed a point P(x, y)

corresponding to a given complex number (x, y). This means that the first coordinate x and the second coordinate y in the ordered pair denote the coordinates of the point corresponding to $z = x + iy$. We call this plane the complex plane or the z-plane and the X, Y axes as the real and imaginary axes respectively. In the discussion that follows, a point in a plane and a complex number are used in the same sense. For example the complex number $2 + 3i$ denotes the point P(2, 3) and $(-2 + 2i)$ denotes a complex number Q(−2, 2). These points are shown in Fig. 1.2.

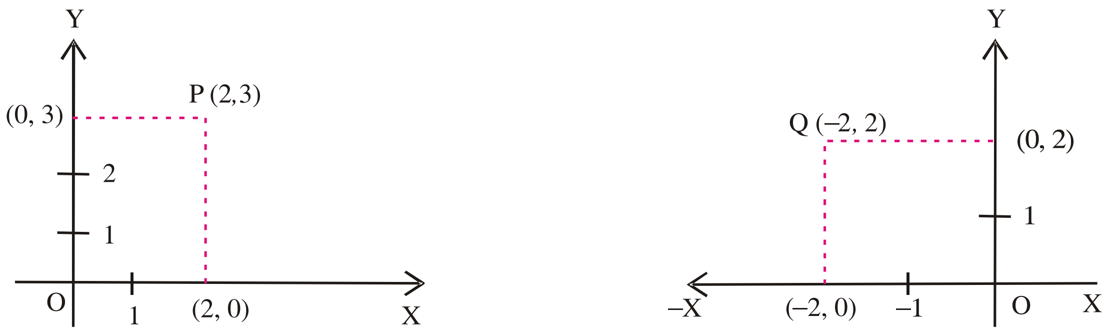


Fig. 1.2

When complex numbers are represented in this way on a complex plane, we call that plane, the Argand plane. In the complex plane any point other than the origin O(0, 0) denotes a vector. It is nothing but the directed line segment joining the origin and the given point. In particular if the point coincides with the origin, it results in a null vector. From this it follows that complex numbers can be represented by vectors in a plane.

1.4.2 Finding the sum and difference of two given complex numbers through Argand diagram

If we take two complex numbers in the Cartesian form (for example, $z_1 = x_1 + i\, y_1,\ z_2 = x_2 + i\, y_2$), their sum or difference can be found geometrically through an Argand diagram.

Let the two given complex numbers z_1, z_2 denote the two points P_1, P_2 on the complex plane (Fig. 1.3). Let the coordinates of P_1 be $z_1 : (x_1, y_1)$ and that of P_2 be $z_2 : (x_2, y_2)$. By associating P_1 with z_1 and P_2 with z_2, we determine the points represented by $(z_1 + z_2), (z_1 - z_2)$ geometrically, employing parallelogram law.

If the origin O (0, 0) is joined to P_1, P_2 and P_2R is drawn through P_2 in such a way that it is parallel and equal to OP_1, then the coordinates of R will be $(x_1 + x_2, y_1 + y_2)$. Therefore the point R denotes the complex number $(z_1 + z_2)$. We write this in the vector form $\mathbf{OR} = \mathbf{OP}_1 + \mathbf{P}_1\mathbf{R} = \mathbf{OP}_1 + \mathbf{OP}_2 = z_1 + z_2$. Hence R denotes the complex number $z_1 + z_2$.

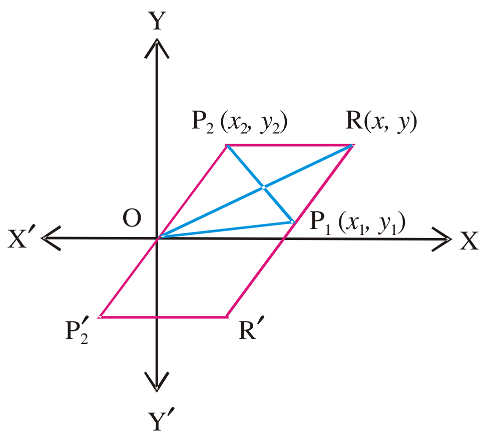


Fig. 1.3

In a similar way, as shown in Fig. 1.3, if P_2O is extended to P_2' such that $P_2O = OP_2'$, the coordinates of P_2' become $(-x_2, -y_2)$. As such we denote P_2' with the complex number $(-x_2, -y_2)$. Adopting a similar procedure, as was done in the case of addition, let the parallelogram $OP_2'R'P_1$ be completed.

Then the point R' denotes the complex number $z_1 + (-z_2) = z_1 - z_2$.

1.4.3 Finding product and quotient of two complex numbers geometrically

Let us first find the product of two complex numbers $z_1 \cdot z_2$. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + i\,y_2$ be the points P_1 and P_2 in the complex plane. For convenience, take $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

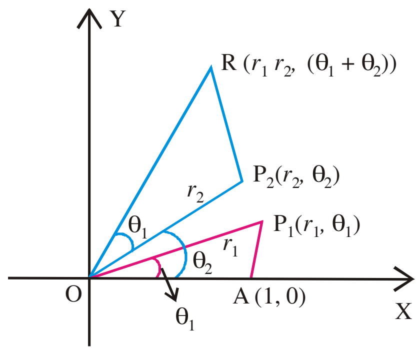


Fig. 1.4

Then it follows that $OP_1 = r_1$, $OP_2 = r_2$.

Now take $A(1, 0)$ on the real axis. Since $\angle AOP_1 = \theta_1$, $\angle AOP_2 = \theta_2$, construct $\Delta\, OP_2R$ similar to $\Delta\, OAP_1$, as shown in Fig. 1.4.

Since $\Delta\, OAP_1$, $\Delta\, ORP_2$ are similar, $\dfrac{OA}{OP_2} = \dfrac{OP_1}{OR}$

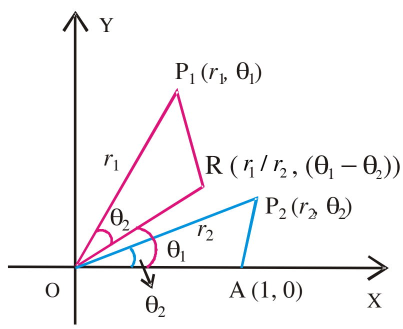


Fig. 1.5

i.e., $OP_1 \cdot OP_2 = OA \cdot OR$ i.e., $r_1 \cdot r_2 = 1.OR$ or $OR = r_1 r_2$. Also $\angle AOR = \angle AOP_2 + \angle ROP_2 = \theta_2 + \theta_1$. Hence the polar coordinates of R are $(r_1 r_2, \theta_1 + \theta_2)$. This means $r_1\, r_2\, cis(\theta_1 + \theta_2)$ denotes the point R. Hence R denotes the complex number $z_1 z_2$.

We shall now find the quotient (z_1 / z_2) of two complex numbers z_1 and z_2 when $z_2 \neq 0$. Following the discussion made on the product, take $z_1 = x_1 + iy_1 = r_1\, cis\, \theta_1$, $z_2 = x_2 + iy_2 = r_2\, cis\, \theta_2$ and denote z_1, z_2 with p_1, p_2 respectively.

From Fig. 1.5, $OP_1 = r_1$; $OP_2 = r_2$; $\angle XOP_1 = \theta_1$, $\angle XOP_2 = \theta_2$. Denote the point $(1, 0)$ on OX by A. Construct $\Delta\, ORP_1$ similar to $\Delta\, OAP_2$, as shown in the Fig. 1.5. Then $\dfrac{OA}{OR} = \dfrac{OP_2}{OP_1}$, i.e., $\dfrac{1}{OR} = \dfrac{r_2}{r_1}$

or $OR = \dfrac{r_1}{r_2}$; $\angle AOR = \angle AOP_1 - \angle P_1OR = \theta_1 - \theta_2$.

$$\therefore \frac{r_1}{r_2}\, cis\,(\theta_1 - \theta_2) \text{ denotes the point R. Hence } R = \frac{z_1}{z_2}.$$

In the examples that follow, we illustrate the locus of the geometrical aspects of the complex number.

1.4.4 Note : Geometrical meanings

(i) If $z = a + bi$. Then $\bar{z} = a - bi$ represents the mirror image of the point $z = a + bi$ about the X-axis.

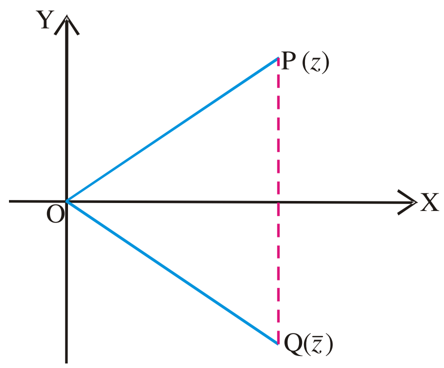


Fig. 1.6

(ii) $|z_1 - z_2|$

Consider two points $A(z_1)$ and $B(z_2)$ in the Argand plane such that $z_1 = x_1 + iy_1 = A(x_1, y_1)$ and $z_2 = x_2 + iy_2 = B(x_2, y_2)$.

$$\text{Then } |z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = AB$$

i.e., the distance between AB is $|z_1 - z_2|$.

(iii) Perpendicular bisector

Let $P(z)$ be any point on the perpendicular bisector of AB, then PA = PB.

$\Rightarrow |z - z_1| = |z - z_2|$

which is the required equation of the perpendicular bisector of the line segment joining $A(z_1)$ and $B(z_2)$.

Note that similarly we can prove the following results.

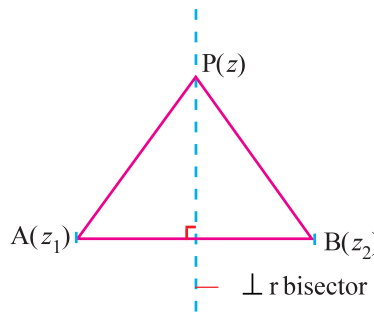


Fig. 1.7

(iv) Equation of straight line joining two points $A(z_1)$ and $B(z_2)$ is

$$\text{Arg}\left(\frac{z - z_1}{z - z_2}\right) = 0 \text{ or } \pi.$$

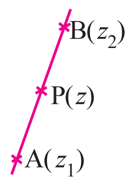


Fig. 1.8

(v) Equation of circle

Let $P(z)$ be any point on a circle with centre at $C(z_0)$ and radius r

then $CP = r \Rightarrow |z - z_0| = r$, which is the required equation of circle.

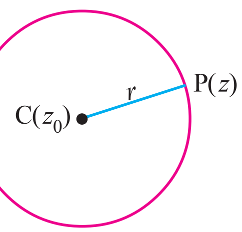


Fig. 1.9

(vi) Diameter form of a circle

$$\operatorname{Arg}\left(\frac{z-z_1}{z-z_2}\right) = \pm\frac{\pi}{2}$$

represents a circle with end points of a diameter as z_1 and z_2.

Its another form is

$|z-z_1|^2 + |z-z_2|^2 = |z_1-z_2|^2.$

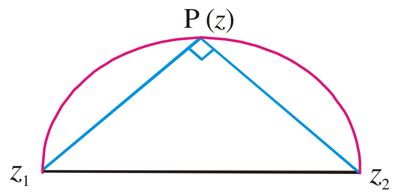


Fig. 1.10

(vii) Consider the equation $\left|\dfrac{z-z_1}{z-z_2}\right| = k.$

The locus of a point z is a circle, if $k \neq 1$ and a straight line, if $k = 1$.

(viii) Equation of ellipse.

We know that the sum of the distances of any point on ellipse from two fixed points (focii) is constant and $2a$.

$\therefore$ The equation of ellipse with focii at z_1 and z_2 is $|z-z_1| + |z-z_2| = 2a$, where $2a > |z_1 - z_2|$ and $2a$ is the length of major axis.

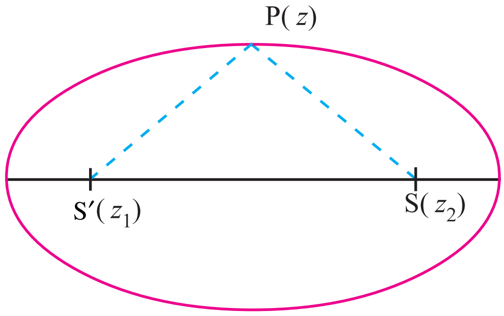


Fig. 1.11

(ix) Equation of hyperbola.

We know that the difference of the distances of any point on hyperbola from two fixed points (focii) is constant and $2a$.

$\therefore$ The equation of hyperbola with focii at z_1 and z_2 is $|z-z_1| - |z-z_2| = 2a$, where $0 < 2a < |z_1 - z_2|$.

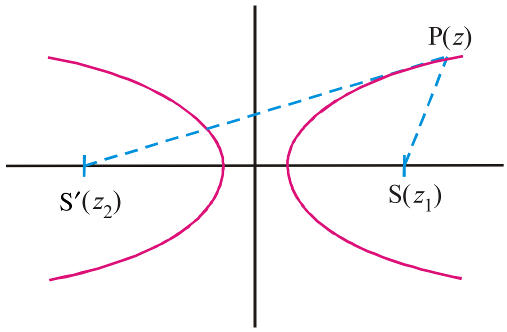


Fig. 1.12

1.4.5 Solved Problems

1. Problem: *Show that the points in the Argand diagram represented by the complex numbers* $1+3i,\ 4-3i,\ 5-5i$ *are collinear.*

Solution : Let the three complex numbers be represented in the Argand plane by the points P, Q, R respectively. Then P = (1, 3), Q = (4, −3) and R = (5, −5). The slope of the line segment joining P, Q is $\dfrac{3+3}{1-4} = \dfrac{6}{-3} = -2$. Similarly the slope of the line segment joining Q, R is $\dfrac{-3+5}{4-5} = \dfrac{2}{-1} = -2$. Since the slope of PQ is the slope of QR, the points P, Q, R are collinear.

2. Problem: *Find the equation of the straight line joining the points represented by* $(-4+3i),\ (2-3i)$ *in the Argand plane.*

Solution : Take the given points as $A = -4+3i = (-4, 3),\ B = 2-3i = (2, -3)$. Then the equation of the straight line $\overline{AB}$ is $y - 3 = \dfrac{3+3}{-4-2}(x+4)$

i.e., $x + y + 1 = 0$.

3. Problem: $z = x + iy$ *represents a point in the Argand plane. Find the locus of* z *such that* $|z| = 2$.

Solution: Let $z = x+iy$. Then $|z| = 2$ if and only if $|x+iy| = 2$ if and only if $\sqrt{x^2+y^2} = 2$ if and only if $x^2 + y^2 = 4$.

$x^2 + y^2 = 4$ represents a circle with centre at $(0, 0)$ and radius 2.

$\therefore$ The locus of $|z| = 2$ is the circle $x^2 + y^2 = 4$.

4. Problem: *The point* P *represents a complex number z in the Argand plane. If the amplitude of* z *is* $\dfrac{\pi}{4}$, *determine the locus of* P.

Solution: Let $z = x+iy$. By hypothesis, amplitude of $z = \pi/4$.

Hence $x = |z|\cos\dfrac{\pi}{4} = \dfrac{|z|}{\sqrt{2}}$ and $y = |z|\sin\dfrac{\pi}{4} = \dfrac{|z|}{\sqrt{2}}$.

Hence $x \ge 0,\ y \ge 0$ and $x = y$. Clearly for any $x \in [0, \infty)$, the point $x + ix$ has amplitude $\dfrac{\pi}{4}$.

$\therefore$ The locus of P is the ray $\left\{(x, y) \in \mathbf{R}^2 \;\middle|\; x \ge 0,\ y \ge 0,\ x = y\right\}$.

5. Problem: *If the point* P *denotes the complex number* $z = x+iy$ *in the Argand plane and if* $\dfrac{z-i}{z-1}$ *is a purely imaginary number, find the locus of* P.

Solution: We note that $\dfrac{z-i}{z-1}$ is not defined if $z = 1$.

Since $z = x + iy$, $\dfrac{z-i}{z-1} = \dfrac{x+iy-i}{x+iy-1} = \dfrac{x+i(y-1)}{x-1+iy}$

$$= \frac{[x+i(y-1)][(x-1)-iy]}{[(x-1)+iy][(x-1)-iy]} = \frac{x^2+y^2-x-y}{(x-1)^2+y^2} + i\left(\frac{1-x-y}{(x-1)^2+y^2}\right)$$

$\dfrac{z-i}{z-1}$ will be purely imaginary, if $z \ne 1$ and $\dfrac{x^2+y^2-x-y}{(x-1)^2+y^2} = 0$

i.e., $x^2+y^2-x-y = 0$ and $(x, y) \ne (1, 0)$.

$\therefore$ The locus of P is the circle $x^2+y^2-x-y=0$ excluding the point $(1, 0)$.

6. Problem: *Describe geometrically the following subsets of* $\mathbf{C}$:

(i) $\{z \in \mathbf{C} \mid |z-1+i| = 1\}$ (ii) $\{z \in \mathbf{C} \mid |z+i| \le 3\}$

Solution

(i) Let $S = \{z \in \mathbf{C} \mid |z-1+i| = 1\}$

If we write $z=(x,\ y)$, then $S=\{(x,\ y)\in \mathbf{R}^2 \mid |x+iy-1+i|=1\}$

$$=\{(x,\ y)\in \mathbf{R}^2 \mid |(x-1)+i(y+1)|=1\}$$

$$=\{(x,\ y)\in \mathbf{R}^2 \mid (x-1)^2+(y+1)^2=1\}$$

Hence S is a circle with centre $(1,\ -1)$ and radius 1 unit.

(ii) Let $S'=\{z\in \mathbf{C} \mid |z+i|\le 3\}$

Then $S'=\{(x,\ y)\in \mathbf{R}^2 \mid |x+iy+i|\le 3\}$

$$=\{(x,\ y)\in \mathbf{R}^2 \mid |x+i(y+1)|\le 3\}$$

$$=\{(x,\ y)\in \mathbf{R}^2 \mid x^2+(y+1)^2\le 9\}$$

Hence S' is the closed circular disc with centre at $(0,\ -1)$ and radius 3 units.

Exercise 1(d)

I. 1. (i) Find the equation of the perpendicular bisector of the line segment joining the points $7+7i, 7-7i$ in the Argand plane.

(ii) Find the equation of the straight line joining the points $-9+6i, 11-4i$ in the Argand plane.

2. If $z=x+iy$ and if the point P in the Argand plane represents z, then describe geometrically the locus of z satisfying the equations

(i) $|z-2-3i|=5$ (ii) $2|z-2|=|z-1|$

(iii) $\mathrm{Img}\ z^2=4$ (iv) $\mathrm{Arg}\left(\dfrac{z-1}{z+1}\right)=\dfrac{\pi}{4}$

3. Show that the points in the Argand diagram represented by the complex numbers $2+2i,\ -2-2i,$ $-2\sqrt{3}+2\sqrt{3}i$ are the vertices of an equilateral triangle.

4. Find the eccentricity of the ellipse whose equation is $|z-4|+\left|z-\dfrac{12}{5}\right|=10$.

II. 1. If $\dfrac{z_3-z_1}{z_2-z_1}$ is a real number, show that the points represented by the complex numbers $z_1,\ z_2,\ z_3$ are collinear.

2. Show that the four points in the Argand plane represented by the complex numbers $2+i,\ 4+3i,$ $2+5i,\ 3i$ are the vertices of a square.

3. Show that the points in the Argand plane represented by the complex numbers $-2+7i,\ \dfrac{-3}{2}+\dfrac{1}{2}i,\ 4-3i,\ \dfrac{7}{2}(1+i)$ are the vertices of a rhombus.

4. Show that the points in the Argand diagram represented by the complex numbers $z_1,\ z_2,\ z_3$ are collinear if and only if there exist three real numbers p, q, r not all zero, satisfying $p z_1+q z_2+r z_3=0$ and $p+q+r=0$.

5. The points P, Q denote the complex numbers z_1, z_2 in the Argand diagram. O is the origin. If $z_1\,\bar{z}_2 + \bar{z}_1\,z_2 = 0$, then show that $\underline{|POQ} = 90^0$.

6. The complex number z has argument $\theta, 0 < \theta < \dfrac{\pi}{2}$ and satisfy the equation $|z - 3i| = 3$. Then prove that $\left(\cot\theta - \dfrac{6}{z}\right) = i$.

Key Concepts

- A Complex number is an ordered pair of real numbers. It is denoted by (a, b), $a \in \mathbf{R}$, $b \in \mathbf{R}$.
- Two complex numbers $z_1 = (a, b)$ and $z_2 = (c, d)$ are said to be equal if and only if $a = c$ and $b = d$.
- Algebra of complex numbers : If $z_1 = (a, b)$, $z_2 = (c, d)$ then
  $z_1 + z_2 = (a + c, b + d)$; $z_1 - z_2 = (a - c, b - d)$
  $z_1 \,.\, z_2 = (ac - bd, ad + bc)$; $\dfrac{z_1}{z_2} = \left(\dfrac{ac + bd}{c^2 + d^2}, \dfrac{bc - ad}{c^2 + d^2}\right)$
- Representation of a complex number
  $z = (a, b)$ is represented by $(a + ib)$, a is called the real part, b is called the imaginary part and $i = (0, 1)$.
- Conjugate of a complex number
  For any complex number $z = a + bi$, the conjugate of z, denoted by $\bar{z}$ is defined as $a + (-b)\,i$ and we write $\bar{z} = a - ib$.
- The modulus of a complex number $z = x + iy$ is defined as a non-negative real number $r = \sqrt{x^2 + y^2}$. It is denoted by $|z|$ and we write mod z.
- Any real number θ satisfying the equations $\cos\theta = \dfrac{x}{r}$, $\sin\theta = \dfrac{y}{r}$ is called an amplitude or argument of z. The unique argument θ of z satisfying $-\pi < \theta \le \pi$ is called the principal argument of z and is denoted by Arg z.
- The polar form or modulus – amplitude form of the complex number $z = (x, y)$ is $r(\cos\theta + i\sin\theta)$.

Historical Note

It was Greek mathematicians who realised first that the square root of a negative real number does not exist in the real number system. But it was the Indian mathematician *Mahavira* (850 A.D) who stated this difficulty very clearly in his work 'Ganitha Sara Sangraha' : "As in the nature of things a negative (quantity) is not a square (quantity) – it has, therefore, no square root". *Bhaskaracharya*, another Indian mathematician in 1150 A.D. wrote in his work 'Bijaga Nitha' : "There is no square root of a negative quantity, for, it is not a square". *Cardan* (1545 A.D.), in solving the equations $x + y = 10$, $xy = 40$ obtained $x = 5 + \sqrt{-15}$ and $y = 5 - \sqrt{-15}$, but discarded these values as solution by saying that these numbers are 'useless'. But *Euler* (1707 - 1783) was the first to introduce the symbol i for $\sqrt{-1}$ and it was *Hamilton* (1805 - 1865) who regarded the complex number as an ordered pair of real numbers – thus giving a more general definition, which avoids the use of the so called 'imaginary numbers' in mathematics.

Answers

Exercise 1(a)

I. 1. $(-4, -4)$ 2. $(-24, 28)$ 3. (i) $(-\sqrt{3}\ -5)$, (ii) $(-4, -1)$, (iii) $(14, -8)$

II. 1. $\left(\frac{9}{5}, \frac{12}{5}\right)$ 2. $(0, 2\sin\theta)$

3. (i) $\left(\frac{3}{25}, \frac{-4}{25}\right)$ (ii) $(\sin\theta, -\cos\theta)$ (iii) $\left(\frac{7}{625}, \frac{-24}{625}\right)$ (iv) $\left(-\frac{2}{5}, \frac{-1}{5}\right)$

Exercise 1(b)

I. 1. (i) $18 + i(-1)$ (ii) $-11 + i(-2)$ (iii) $\frac{a^2-b^2}{a^2+b^2} + i\left(\frac{-2ab}{a^2+b^2}\right)$ (iv) $\frac{86}{325} + \frac{27}{325}i$

(v) $(-6+\sqrt{2}) + i(\sqrt{3}+2\sqrt{6})$ (vi) $\frac{5}{8} + i(0)$ (vii) $2 + i(0)$

(viii) $0 + i.1$ (ix) $0 + i.1$ (x) $14 + i.28$ (xi) $\frac{-8}{13} + i.(0)$

2. (i) $3 - 4i$ (ii) $11 - 23i$ (iii) $-38 + 8i$ (iv) $\frac{1-7i}{10}$

3. (i) -1 (ii) $1 + 3i$

4. (i) $\pm(4+3i)$ (ii) $\pm(1-3i)$ (iii) $\pm(2+i)$ (iv) $\pm(1+4i\sqrt{3})$

5. (i) $\frac{\sqrt{5}-3i}{14}$ (ii) i (iii) $-i$

II. 1. (i) $(a^2+b^2)^2$ (iv) $\dfrac{2(x+3)+y}{(x+3)^2+y^2}, \dfrac{x-2y+3}{(x+3)^2+y^2}$

3. (ii) $\dfrac{8ab(a^2-b^2)i}{(a^2+b^2)^2}$ (iii) $x=3, y=-1$

4. (i) $x=4$ (ii) $x=0,\ y=-2$ (iii) (a) $\theta=n\pi, n\in I$ (b) $\theta = 2n\pi \pm \dfrac{\pi}{3}, n\in I$

(iv) $x=-4,\ y=6$

Exercise 1(c)

I. 1. (i) $\sqrt{2}\left[\cos\left(-\dfrac{\pi}{4}\right)+i\sin\left(-\dfrac{\pi}{4}\right)\right]$ (ii) $2\left[\cos\left(\dfrac{\pi}{3}\right)+i\sin\left(\dfrac{\pi}{3}\right)\right]$

(iii) $2(\cos 150^0 + i\sin 150^0)$ (iv) $2\left[\cos\left(-\dfrac{2\pi}{3}\right)+i\sin\left(-\dfrac{2\pi}{3}\right)\right]$

2. (i) $8(4+3i), 40$ 3. (i) $\begin{cases} 0, & \text{if Arg } z \neq \pi \\ 2\pi, & \text{if Arg } z = \pi \end{cases}$ (ii) $\dfrac{\pi}{2}$ (iii) $\dfrac{\pi}{2}$

4. (i) $\theta = 2(\alpha+\beta)$ (ii) $\dfrac{\pi}{6}$ radians (iii) 1 (iv) 1

5. (i) $x^2+y^2=1$ (ii) $x=1, y>0$ (iii) $\dfrac{2\pi}{15}$ (iv) 0^0

II. 1. (i) $2-i; \sqrt{5}$ (ii) $\dfrac{2+2i}{5}, \dfrac{2\sqrt{2}}{5}$

2. (ii) $x-y=1$ (iii) $x^2+y^2-6x+2y-6=0$ (iv) $\mathbf{y}=0$

3. (i) $x^2+y^2-3x-10=0$; a circle with centre at $\left(\dfrac{3}{2}, 0\right)$ and radius $\dfrac{7}{2}$ units

(ii) $x^2+y^2-4x-8=0$; a circle with centre at $(2, 0)$ and radius $2\sqrt{3}$ units

(iii) $2y-1=0$; a line parallel to x-axis.

(iv) $25x^2+9y^2=225$; an ellipse with centre $(0, 0)$ with eccentricity $\dfrac{4}{5}$ and major axis is parallel to Y-axis.

4. (ii) Y-axis

Exercise 1(d)

I. 1. (i) $y=0$ (ii) $x+2y-3=0$

2. (i) $x^2+y^2-4x-6y-12=0$

(ii) $x^2+y^2-6x+7=0$, circle with centre $(3, 0)$ and radius 1 unit.

(iii) $xy=2$, a rectangular hyperbola.

(iv) $x^2+y^2-2y-1=0$

Chapter 2

De Moivre's Theorem

"In any problem there must be an unknown - if every thing is known, there is nothing to seek, nothing to do"

– G . Polya

De Moivre

(1667-1754)

Abraham De Moivre *was a French mathematician famous for De Moivre's formula, which links complex numbers and trigonometry, and for his work on the normal distribution and probability theory. He pioneered the development of Analytic Geometry. De Moivre wrote a book on probability theory, entitled 'The Doctrine of Chances'.*

Introduction

In the previous chapter we learnt that $\text{cis}\theta_1 \,.\, \text{cis}\,\theta_2 = \text{cis}(\theta_1 + \theta_2)$ and hence $(\text{cis}\,\theta)^2 = \text{cis}\,2\theta$. In this chapter we extend this result for any integer. This extension is called De Moivre's theorem for integral indices. We use it in studying the n^{th} roots of unity. We also obtain a version of De Moivre's theorem for rational indices.

2.1 De Moivre's Theorem - Integral and Rational Indices

In this section De Moivre's theorem is proved. By using this theorem all the n^{th} roots of a complex number $z \neq 0$ are determined. As a particular case, all the n^{th} roots of unity are determined and their geometrical representations are obtained.

2.1.1 Theorem (De Moivre's theorem for integral index)

For any real number θ *and any integer* n*,* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Proof: Let θ be a given real number. We distinguish three cases.

Case (i) Let n be a positive integer. We prove the theorem using the principle of mathematical induction on n.

Let $S(n)$ be the statement $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

If $n = 1$, then L.H.S. $= (\cos\theta + i\sin\theta) = \cos 1\theta + i\sin 1\theta =$ R.H.S., hence $S(1)$ is true. Assume that $S(k)$ is true for $k \in \mathbf{N}$.

i.e., $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$.

Multiplying both the sides of the above equation with $(\cos\theta + i\sin\theta)$ we get

$(\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$.

$$
\begin{aligned}
(\cos\theta + i\sin\theta)^{k+1} &= \cos k\theta\cos\theta + i\sin k\theta\cos\theta + i\cos k\theta\sin\theta + i^2\sin k\theta\sin\theta\\
&= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)\\
&= \cos(k\theta + \theta) + i\sin(k\theta + \theta) = \cos(k+1)\theta + i\sin(k+1)\theta = \text{R.H.S.}
\end{aligned}
$$

$\therefore S(k+1)$ is true

By the principle of mathematical induction, $S(n)$ is true for all positive integers n. i.e., $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all $n \in \mathbf{Z}^+$.

Case (ii) Let $n = 0$ then L.H.S. $= (\cos\theta + i\sin\theta)^0 = 1 = \cos 0\theta + i\sin 0\theta =$ R.H.S.

Hence $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ in this case also.

Case (iii) Let n be a negative integer and $n = -m$, where $m \in \mathbf{Z}^+$.

$$
\begin{aligned}
\therefore \text{L.H.S.} = (\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m} &= \frac{1}{(\cos\theta + i\sin\theta)^m}\\
&= \frac{1}{\cos m\theta + i\sin m\theta} \quad \text{(from case (i))}\\
&= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta - i^2\sin^2 m\theta}\\
&= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}\\
&= \cos(-m)\theta + i\sin(-m)\theta\\
&= \cos n\theta + i\sin n\theta = \text{R.H.S.}
\end{aligned}
$$

2.1.2 Note

(i) It is customary to write $cis\,\theta$ for $\cos\theta + i\sin\theta$. Thus we may state the De Moivre's theorem as $(cis\,\theta)^n = cis\,(n\,\theta)$ if $n \in \mathbf{Z}$.

(ii) $(\cos\theta + i\sin\theta)^{-n} = \cos(-n)\theta + i\sin(-n)\theta = \cos n\theta - i\sin n\theta$, provided '$n$' is an integer.

(iii) $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta - i^2\sin^2\theta = \cos^2\theta + \sin^2\theta = 1.$

$$\therefore (\cos\theta + i\sin\theta) = \frac{1}{\cos\theta - i\sin\theta}, \quad (\cos\theta - i\sin\theta) = \frac{1}{\cos\theta + i\sin\theta}.$$

(iv) $(\cos\theta - i\sin\theta)^n = \left(\dfrac{1}{\cos\theta + i\sin\theta}\right)^n = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$, provided '$n$' is an integer.

(v) $cis\,\theta . cis\,\phi = cis(\theta + \phi)$ for any $\theta, \phi \in \mathbf{R}$.

2.1.3 Solved Problems

1. Problem: *Simplify* $\dfrac{(\cos\alpha + i\sin\alpha)^4}{(\sin\beta + i\cos\beta)^8}$

Solution:
$$\frac{(\cos\alpha + i\sin\alpha)^4}{(\sin\beta + i\cos\beta)^8} = \frac{(\cos\alpha + i\sin\alpha)^4}{(-i^2\sin\beta + i\cos\beta)^8} = \frac{(\cos\alpha + i\sin\alpha)^4}{(i)^8(\cos\beta - i\sin\beta)^8}$$
$$= (\cos\alpha + i\sin\alpha)^4(\cos\beta - i\sin\beta)^{-8} \quad \left[\because i^8 = (i^2)^4 = (-1)^4 = 1\right]$$
$$= (\cos 4\alpha + i\sin 4\alpha)(\cos 8\beta + i\sin 8\beta)$$
$$= \cos(4\alpha + 8\beta) + i\sin(4\alpha + 8\beta).$$

2. Problem: *If m, n are integers and $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$ then prove that $x^m y^n + \dfrac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$ and $x^m y^n - \dfrac{1}{x^m y^n} = 2\,i\sin(m\alpha + n\beta)$.*

Solution: $x^m = (\cos\alpha + i\sin\alpha)^m = \cos m\alpha + i\sin m\alpha$

$y^n = (\cos\beta + i\sin\beta)^n = \cos n\beta + i\sin n\beta$

$\therefore x^m y^n = (\cos m\alpha + i\sin m\alpha)(\cos n\beta + i\sin n\beta)$

$$= \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta) \quad \text{... (1)}$$

$$\frac{1}{x^m y^n} = \frac{1}{\cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)}$$
$$= \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta) \quad \text{... (2)}$$

By adding (1) and (2), we get $x^m y^n + \dfrac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$

By subtracting (2) from (1), we get $x^m y^n - \dfrac{1}{x^m y^n} = 2\,i\sin(m\alpha + n\beta)$.

3. Problem: *If n is a positive integer,*

show that $(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}} \cos\left(\dfrac{n\pi}{4}\right)$

Solution: $(1+i) = \sqrt{2}\left(\dfrac{1}{\sqrt{2}} + i\,\dfrac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

$(1-i) = \sqrt{2}\left(\dfrac{1}{\sqrt{2}} - i\,\dfrac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\dfrac{\pi}{4} - i\sin\dfrac{\pi}{4}\right)$

$$(1+i)^n = \left(\sqrt{2}\right)^n \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) \quad \ldots (1)$$

$$(1-i)^n = \left(\sqrt{2}\right)^n \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right) \quad \ldots (2)$$

By adding (1) and (2), we get $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}}\left(2\cos\dfrac{n\pi}{4}\right) = 2^{\frac{n+2}{2}}\cos\dfrac{n\pi}{4}$.

4. Problem: *If n is an integer then show that* $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n$

$$= 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right).$$

Solution: L.H.S. $= (1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n$

$$= (2\cos^2\frac{\theta}{2} + 2\,i\sin\frac{\theta}{2}\cos\frac{\theta}{2})^n + (2\cos^2\frac{\theta}{2} - 2\,i\sin\frac{\theta}{2}\cos\frac{\theta}{2})^n$$

$$= 2^n\cos^n\frac{\theta}{2}\left[\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^n + \left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)^n\right]$$

$$= 2^n\cos^n\frac{\theta}{2}\left[\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2} + \cos\frac{n\theta}{2} - i\sin\frac{n\theta}{2}\right]$$

$$= 2^n\cos^n\frac{\theta}{2}\left(2\cos\frac{n\theta}{2}\right) = 2^{n+1}\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2} = \text{R.H.S.}$$

5. Problem: *If* $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, *Prove that*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2} = \sin^2\alpha + \sin^2\beta + \sin^2\gamma.$$

Solution: $(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma)$

$$= (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0 + i0$$

$$(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma) = 0 \qquad \text{... (1)}$$

Let $x = cis\,\alpha$, $y = cis\,\beta$, $z = cis\,\gamma$ then $x + y + z = 0$ (by (1)), then

$$x^2 + y^2 + z^2 = -2(xy + yz + zx) = -2xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$$

$$= -2xyz\,[\cos\alpha - i\sin\alpha + \cos\beta - i\sin\beta + \cos\gamma - i\sin\gamma]$$

$$= -2xyz\,[(\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)]$$

$$= -2xyz(0 - i0) = 0.$$

$$\therefore x^2 + y^2 + z^2 = 0$$

$$\Rightarrow (\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$$

$$\Rightarrow \cos 2\alpha + i\sin 2\alpha + \cos 2\beta + i\sin 2\beta + \cos 2\gamma + i\sin 2\gamma = 0$$

$$\Rightarrow (\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + i(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0$$

$$\therefore \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$$

$$2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1 = 0$$

$$2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 3$$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$$

$$1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = \frac{3}{2}$$

$$\therefore \sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}.$$

Exercise 2(a)

I. 1. If n is an integer then show that $(1+i)^{2n}+(1-i)^{2n} = 2^{n+1}\cos\frac{n\pi}{2}$.

2. Find the values of the following:

(i) $\left(1+i\sqrt{3}\right)^3$ (ii) $(1-i)^8$

(iii) $(1+i)^{16}$ (iv) $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 - \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$

II. 1. If α, β are the roots of the equation $x^2-2x+4=0$ then for any $n \in \mathbf{N}$ show that $\alpha^n+\beta^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$.

2. If $\cos\alpha+\cos\beta+\cos\gamma = 0 = \sin\alpha+\sin\beta+\sin\gamma$ then show that

(i) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

(iii) $\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)=0$

3. If n is an integer and $z = cis\,\theta, \left(\theta \neq (2n+1)\frac{\pi}{2}\right)$, then show that $\frac{z^{2n}-1}{z^{2n}+1} = i\tan n\theta$.

4. If $(1+x)^n = a_0+a_1x+a_2x^2+....+a_nx^n$, then show that

(i) $a_0-a_2+a_4-a_6+.... = 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$

(ii) $a_1-a_3+a_5+.... = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}$

2.2 n^{th} roots of unity - Geometrical Interpretations - Illustrations

When z_0 is a positive real number there is one and only one positive real number r such that $r^n = z_0$, which we denote by $z_0^{1/n}$. However there are many more complex numbers 'ω' such that $\omega^n = z_0$ For example $cis\,0, cis\frac{2\pi}{3}, cis\frac{4\pi}{3}$ satisfy $\omega^3 = 1$. The following theorems tell us how to find all the n^{th} roots of a non-zero complex number.

2.2.1 Theorem (De Moivre's theorem for rational index)

If n is a rational number, then one of the values of $(\cos\theta + i\sin\theta)^n$ *is* $\cos n\theta + i\sin n\theta$.

Proof : Let $n = \frac{p}{q}$, where p, q are integers and q is a positive integer. Then

$$(\cos n\theta + i\sin n\theta)^q = \left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q = (\cos p\theta + i\sin p\theta) \text{[from theorem 2.1.1]}$$

$$= (\cos\theta + i\sin\theta)^p$$

$\therefore$ Hence, one of the q^{th} roots of $(\cos\theta + i\sin\theta)^p$ is $\cos n\theta + i\sin n\theta$.

i.e., one of the values of $(\cos\theta + i\sin\theta)^{p/q}$ is $\left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)$

Hence it follows that $(\cos n\theta + i\sin n\theta)$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

2.2.2 Definition (n^{th} root of a complex number)

Let n be a positive integer and $z_0 \neq 0$ be a given complex number. Any complex number z satisfying $z^n = z_0$ is called an n^{th} root of z_0 and is denoted by $z_0^{1/n}$ or $\sqrt[n]{z_0}$.

Now we learn how to find the n^{th} roots of a given complex number z_0, using geometric techniques.

2.2.3 Theorem: *Let $z_0 = r_0(\cos\theta_0 + i\sin\theta_0) \neq 0$ and 'n' be a positive integer. For $k \in \{0, 1, 2, ..., (n-1)\}$, let $a_k = r_0^{1/n}\, cis\left(\frac{\theta_0 + 2k\pi}{n}\right)$. Then $a_0, a_1, ..., a_{n-1}$ are all the 'n' distinct n^{th} roots of z_0 and any n^{th} root of z_0 coincides with one of them.*

Proof: We divide the proof into three steps

(i) For each $k \in \{0, 1, 2, ..., (n-1)\}$, $a_k^n = z_0$

(ii) If $\omega^n = z_0$, then $\omega = a_k$ for some $k \in \{0, 1, 2, ..., (n-1)\}$

(iii) If $0 \le i \neq j \le (n-1)$ then $a_i \neq a_j$

Step (i) : We have $a_k^n = \left(r_0^{\frac{1}{n}} \, cis \, \frac{\theta_0 + 2k\pi}{n} \right)^n = r_0 \, cis\,(\theta_0 + 2k\pi)$ [by De Moivre's theorem]. Since, the trigonometric functions $\sin\theta, \cos\theta$ are periodic functions of period 2π and k is an integer, $\cos(\theta_0 + 2k\pi) = \cos\theta_0$, $\sin(\theta_0 + 2k\pi) = \sin\theta_0$.

Hence $a_k^n = r_0 \, cis\,(\theta_0) = z_0$.

$$\therefore a_k \text{ is an } n^{th} \text{ root of } z_0.$$

Step (ii) : Let 'ω' be an n^{th} root of z_0. Then $\omega^n = z_0$.

Hence $|\omega|^n = |\omega^n| = |z_0| = r_0 \Rightarrow |\omega| = r_0^{1/n}$

$\therefore \omega = r_0^{1/n} \, cis \, \alpha$ for some $\alpha \in [0, 2\pi)$. Then by 2.1.1

$\omega^n = r_0 \, cis \, n\alpha$ but $\omega^n = z_0 = r_0 \, cis\,(\theta_0)$

$\therefore r_0 \cos n\alpha = r_0 \cos\theta_0$ and $r_0 \sin(n\alpha) = r_0 \sin\theta_0$.

Since $r_0 \neq 0$, we have $\cos(n\alpha) = \cos(\theta_0)$ and $\sin(n\alpha) = \sin\theta_0$.

$\therefore n\alpha = 2m\pi + \theta_0$ for some integer 'm'. From the division algorithm we have $m = nj + k$ for some integer j and for some $k \in \{0, 1, 2, ..., (n-1)\}$

$$\therefore n\alpha = 2nj\pi + 2k\pi + \theta_0$$

$$\therefore \alpha = 2j\pi + \frac{2k\pi + \theta_0}{n}$$

$$\therefore \cos\alpha = \cos\left(\frac{2k\pi + \theta_0}{n}\right), \sin\alpha = \sin\left(\frac{2k\pi + \theta_0}{n}\right)$$

$$\text{Hence } \omega = a_k.$$

Thus any n^{th} root of z_0 coincides with one of $a_0, a_1, a_2, ... a_{n-1}$.

Step (iii) : Let $\eta = r_0^{\frac{1}{n}} \left(\cos\frac{\theta_0}{n} + i \sin\frac{\theta_0}{n} \right)$ and $\xi_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$, $(k = 0, 1, ..., (n-1))$ then ξ_k is the point P_k on the unit circle such that OP_k makes an angle of

$\frac{2\pi k}{n}$ with the positive direction of the x-axis, measured in the positive direction. Since $\frac{2\pi k}{n}\ (k = 0, 1, 2, ..., n-1)$ are 'n' distinct values of $[0, 2\pi)$, it follows that $P_0, P_1, ..., P_{n-1}$ are n-distinct points. Hence $\xi_i \neq \xi_j$ for $i \neq j$. Since $r_0 \neq 0$, $\eta \neq 0$ and $\eta\xi_i \neq \eta\xi_j$, for $i \neq j$. Hence $a_i \neq a_j$ for distinct indices i and j among $0, 1, 2, ..., (n-1)$.

n^{th} roots of unity

2.2.4 Corollary

Let 'n' be a positive integer greater than one and $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, *then* $1, \omega, \omega^2, ..., \omega^{n-1}$ *are all the distinct n^{th} roots of unity.*

Proof : We have $1 = \cos 0 + i\sin 0$. From Theorem 2.2.3 it follows that $a_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\ (k = 0, 1, 2, ..., (n-1))$ are all the 'n' distinct n^{th} roots of unity. From De-Moivre's theorem we have

$$a_k = \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right) = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^k = \omega^k\ (k = 0, 1, 2, ..., (n-1)).$$

2.2.5 Note

(i) Observe that the n^{th} roots of unity differ by an argument $\frac{2\pi}{n}$.

(ii) The sum of the n^{th} roots of unity is zero.

$$\left[\because 1 + \omega + \omega^2 + ... + \omega^{n-1} = \frac{1-\omega^n}{1-\omega} = \frac{1-1}{1-\omega} = 0\right]$$

(iii) Product of the n^{th} roots of the unity is $(-1)^{n-1}$.

$$\left[\because 1.\ \omega.\ \omega^2 .... \omega^{n-1} = \omega^{1+2+3+....+(n-1)} = \omega^{\frac{n(n-1)}{2}} = \left(cis\frac{2\pi}{n}\right)^{\frac{n(n-1)}{2}}\right.$$

$$\left. = cis\,(n-1)\pi = \cos(n-1)\pi + i\sin(n-1)\pi = (-1)^{n-1}\right]$$

(iv) The n^{th} roots of unity $1, \omega, \omega^2, ..., \omega^{n-1}$ are in geometric progression with common ratio ω.

2.2.6 Geometrical representation of the n^{th} roots of unity

We have proved that the n^{th} roots of unity are precisely $cis\,\frac{2k\pi}{n}$ $\left(k = 0, 1, 2, ..., (n-1)\right)$. Clearly all of them lie on the unit circle in the complex plane. Now we locate their positions on the unit circle.

Let us consider the unit circle centred at the origin. For $k \in \{0, 1, 2, ..., (n-1)\}$, let P_k denote the point on the unit circle such that the rays OP_k makes an angle $\frac{2k\pi}{n}$ with the positive direction of the X-axis. (i.e., initial ray) measured in the anti-clock wise sense. Then the polar co-ordiantes of P_k are $\left(1, \frac{2k\pi}{n}\right)$. Hence P_k represents the complex number $cis\,\frac{2\pi k}{n}$. Thus the points $P_0, P_1, P_2, ..., P_{(n-1)}$ represent the n^{th} roots of unity. Note that $P_0, P_1, ..., P_{n-1}$ is a regular n - gon with vertices on the unit circle. The 8th roots of unity are marked in Fig. 2.1.

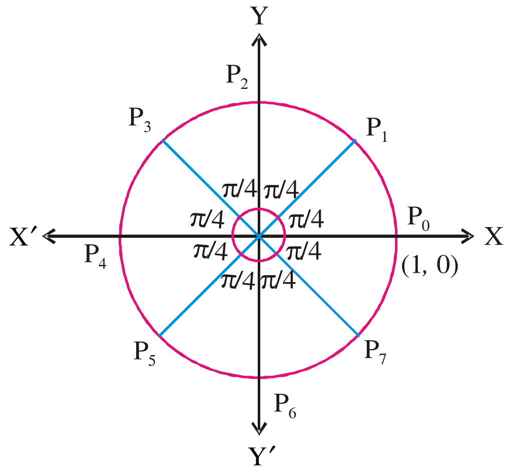


Fig. 2.1

2.2.7 Cube roots of unity

In this section we find all the cube roots of unity and all the cube roots of a positive real number.

2.2.8 Theorem: $1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$ *are the cube roots of the unity.*

Proof: From the corollary 2.2.4 we have the n^{th} roots of unity are $1,\ \omega,\ \omega^2,\ ...,\ \omega^{n-1}$ where $\omega = cis\,\frac{2\pi}{n}$.

$$\therefore \text{ cube roots of unity are } 1,\ \omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2} = \frac{-1+i\sqrt{3}}{2} \text{ and}$$

$$\omega^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{i\sqrt{3}}{2} = \frac{-1-i\sqrt{3}}{2}.$$

2.2.9 Note

1. Cube roots of unity are the vertices of an equilateral triangle which is inscribed in a circle of unit radius and having centre at origin.

2. $1+\omega+\omega^2=0$ and $1\,.\,\omega\,.\,\omega^2=\omega^3=1$.

3. The cube roots of a positive real number 'a' are $a^{1/3}, a^{1/3}\omega, a^{1/3}\omega^2$.

2.2.10 Solved Problems

1. Problem: *Find all the values of* $\left(\sqrt{3}+i\right)^{1/4}$.

Solution: The modulus amplitude form of $\sqrt{3}+i=2\left(\cos 30^0+i\sin 30^0\right)$.

$$\text{Hence } \left(\sqrt{3}+i\right)^{1/4}=\left(2\,cis\frac{\pi}{6}\right)^{1/4}=2^{1/4}\left(cis\frac{2k\pi+\frac{\pi}{6}}{4}\right),\ k=0,1,2,3$$

$$=2^{1/4}\ cis\left(\frac{12k\pi+\pi}{24}\right),\quad k=0,1,2,3$$

$$=2^{1/4}\ cis\,(12k+1)\frac{\pi}{24},\ k=0,1,2,3$$

$\therefore$ All the values of $\left(\sqrt{3}+i\right)^{1/4}$ are $2^{\frac{1}{4}}cis\frac{\pi}{24},\ 2^{\frac{1}{4}}cis\frac{13\pi}{24},\ 2^{\frac{1}{4}}cis\frac{25\pi}{24},\ 2^{\frac{1}{4}}cis\frac{37\pi}{24}$.

2. Problem: *Find all the roots of the equation* $x^{11}-x^7+x^4-1=0$.

Solution: $x^{11}-x^7+x^4-1=x^7\left(x^4-1\right)+1\left(x^4-1\right)=\left(x^4-1\right)\left(x^7+1\right)$

Therefore the roots of the given equations are precisely the 4^{th} roots of unity and 7^{th} roots of -1. They are $cis\frac{2k\pi}{4}=cis\frac{k\pi}{2},\ k\in\{0,1,2,3\}$ and

$$cis\frac{2k\pi+\pi}{7}=cis\,(2k+1)\frac{\pi}{7},\ k\in\{0,1,2,3,4,5,6\}\ [\because -1=cis\,\pi]$$

$\therefore$ The value of x are $cis\,0=1,\ cis\frac{\pi}{2}=i, cis\,\pi=-1, cis\frac{3\pi}{2}=-i,$

$$cis\frac{\pi}{7},\ cis\frac{3\pi}{7},\ cis\frac{5\pi}{7},\ cis\,\pi=-i,\ cis\frac{9\pi}{7},\ cis\frac{11\pi}{7},\ cis\frac{13\pi}{7}$$

i.e., $\pm i, \pm 1,\ cis\frac{\pi}{7},\ cis\frac{3\pi}{7},\ cis\frac{5\pi}{7},\ cis\frac{9\pi}{7},\ cis\frac{11\pi}{7},\ cis\frac{13\pi}{7}$.

3. Problem: *If* $1,\ \omega,\ \omega^2$ *are the cube roots of unity, prove that*

(i) $\left(1-\omega+\omega^2\right)^6+\left(1-\omega^2+\omega\right)^6=128=\left(1-\omega+\omega^2\right)^7+\left(1+\omega-\omega^2\right)^7$

(ii) $(a+b)\left(a\omega+b\omega^2\right)\left(a\omega^2+b\omega\right)=a^3+b^3.$

(iii) $x^2+4x+7=0$ where $x=\omega-\omega^2-2.$

Solution : We use $1+\omega+\omega^2=1+\dfrac{\left(-1+i\sqrt{3}\right)}{2}+\dfrac{\left(-1-i\sqrt{3}\right)}{2}=0$ and

$\omega^3=\left(cis\dfrac{2\pi}{3}\right)^3=cis\,2\pi=1.$

(i) $$\left(1-\omega+\omega^2\right)^6+\left(1-\omega^2+\omega\right)^6=(-\omega-\omega)^6+\left(-\omega^2-\omega^2\right)^6=2^6\left(\omega^6+\omega^{12}\right)$$
$$=2^6(2)=128$$
$$\left(1-\omega+\omega^2\right)^7+\left(1+\omega-\omega^2\right)^7=(-\omega-\omega)^7+\left(-\omega^2-\omega^2\right)^7$$
$$=(-2)^7\left(\theta^7+\theta^{14}\right)=(-2)^7\left(\theta+\theta^2\right)=(-128)(-1)=128$$

(ii) $$(a+b)\left(a\omega+b\omega^2\right)\left(a\omega^2+b\omega\right)=(a+b)\left(a^2\omega^3+ab\omega^4+ab\omega^2+b^2\omega^3\right)$$
$$=(a+b)\left(a^2+ab\left(\omega+\omega^2\right)+b^2\right)=(a+b)\left(a^2-ab+b^2\right)=a^3+b^3$$

(iii) $$x=\omega-\omega^2-2\Rightarrow(x+2)=\omega-\omega^2$$
$$\Rightarrow(x+2)^2=\omega^2+\omega^4-2\omega^3$$
$$\Rightarrow x^2+4x+4=\omega^2+\omega-2=-1-2=-3$$
$$\Rightarrow x^2+4x+7=0.$$

4. Problem: *If* $\alpha,\ \beta$ *are the roots of the equation* $x^2+x+1=0$ *then prove that* $\alpha^4+\beta^4+\alpha^{-1}\beta^{-1}=0$.

Solution: Since α, β are the complex cube roots of unity, we may take $\alpha=\omega,\ \beta=\omega^2$.

$$\therefore\ \alpha^4+\beta^4+\alpha^{-1}\beta^{-1}=\omega^4+\omega^8+\omega^{-1}\omega^{-2}=\omega^3.\omega+\left(\omega^3\right)^2\omega^2+\frac{1}{\omega^3}=\omega+\omega^2+1=0.$$

Exercise 2(b)

I. 1. Find all the values of

(i) $\left(1 - i\sqrt{3}\right)^{1/3}$ (ii) $(-i)^{1/6}$ (iii) $(1+i)^{2/3}$ (iv) $(-16)^{1/4}$ (v) $(-32)^{1/5}$

2. If ABC are angles of a triangle such that $x = \text{cis}\,A$, $y = \text{cis}\,B$, $z = \text{cis}\,C$, then find the value of xyz.

3. (i) If $x = \text{cis}\,\theta$, then find the value of $\left(x^6 + \dfrac{1}{x^6}\right)$

(ii) Find the cube roots of 8.

4. If $1, \omega, \omega^2$ are the cube roots of unity, then prove that

(i) $\dfrac{1}{2+\omega} + \dfrac{1}{1+2\omega} = \dfrac{1}{1+\omega}$

(ii) $(2-\omega)\left(2-\omega^2\right)\left(2-\omega^{10}\right)\left(2-\omega^{11}\right) = 49$

(iii) $(x+y+z)\left(x+y\omega+z\omega^2\right)\left(x+y\omega^2+z\omega\right) = x^3+y^3+z^3-3xyz$

5. Prove that $-\omega$ and $-\omega^2$ are the roots of $z^2 - z + 1 = 0$, where ω and ω^2 are the complex cube roots of unity.

6. If $1, \omega, \omega^2$ are the cube roots of unity, then find the values of the following.

(i) $(a+b)^3 + (a\omega + b\omega^2)^3 + (a\omega^2 + b\omega)^3$

(ii) $(a+2b)^2 + (a\omega^2 + 2b\omega)^2 + (a\omega + 2b\omega^2)^2$

(iii) $(1-\omega+\omega^2)^3$

(iv) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$

(v) $\left(\dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}\right) + \dfrac{(a+b\omega+c\omega^2)}{(b+c\omega+a\omega^2)}$

(vi) $(1+\omega)^3 + (1+\omega^2)^3$

(vii) $(1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5$

II. 1 Solve the following equations.

(i) $x^4 - 1 = 0$ (ii) $x^5 + 1 = 0$ (iii) $x^9 - x^5 + x^4 - 1 = 0$ (iv) $x^4 + 1 = 0$

2. Find the common roots of $x^{12} - 1 = 0$ and $x^4 + x^2 + 1 = 0$.

3. Find the number of 15th roots of unity, which are also 25th roots of unity.

4. If the cube roots of unity are $1, \omega, \omega^2$, then find the roots of the equation $(x-1)^3 + 8 = 0$.

5. Find the product of all the values of $(1+i)^{4/5}$.

6. If $z^2 + z + 1 = 0$, where z is a complex number, prove that

$$\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+\left(z^4+\frac{1}{z^4}\right)^2+\left(z^5+\frac{1}{z^5}\right)^2+\left(z^6+\frac{1}{z^6}\right)^2=12.$$

III. 1. If $1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1}$ be the nth roots of unity, then prove that

$$1^p+\alpha^p+(\alpha^2)^p+(\alpha^3)^p+\ldots+(\alpha^{n-1})^p=\begin{cases}0; \text{ if } p\neq kn \\ n; \text{ if } p=kn\end{cases}, \quad \text{where } p, k \in \mathbf{N}.$$

2. Prove the sum of 99th powers of the roots of the equation $x^7 - 1 = 0$ is zero and hence deduce the roots of $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.

3. If n is a positive integer, show that $(\mathrm{P}+i\mathrm{Q})^{\frac{1}{n}}+(\mathrm{P}-i\mathrm{Q})^{\frac{1}{n}}=2\ (\mathrm{P}^2+\mathrm{Q}^2)^{\frac{1}{2n}}.\cos\left[\frac{1}{n}\tan^{-1}\frac{\mathrm{Q}}{\mathrm{P}}\right]$.

4. Show that one value of $\left[\dfrac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right]^{\frac{8}{3}}$ is -1.

5. Solve $(x-1)^n = x^n$, n is a positive integer.

Key Concepts

- If 'n' is an integer, then $(cis\ \theta)^n = cis\ (n\theta)$ [De-Moivre's theorem for integral index].
- If 'n' is a rational number, then one of the values of $(cis\,\theta)^n$ is $cis\,n\theta$ [De Moivre's theorem for rational index].
- If $z_0 = \gamma_0\, cis\ \theta_0 \neq 0$, then the n^{th} roots of z_0 are $a_k = r_0^{\frac{1}{n}}\ cis\left(\dfrac{2k\pi+\theta_0}{n}\right)$, $k = 0, 1, 2, \ldots., (n-1)$.
- The n^{th} roots of unity are $cis\dfrac{2k\pi}{n}$, $k = 0, 1, 2, 3, \ldots., (n-1)$.
- Cube roots of unity are 1, $\omega = \dfrac{-1+i\sqrt{3}}{2}$, $\omega^2 = \dfrac{-1-i\sqrt{3}}{2}$.

Historical Note

De Moivre formula led Trigonometry to Analysis. He had known the relation $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,\ n \in \mathbf{N}$ for more than twenty years before making it public. *De Moivre* was an intimate friend of *Newton* and revised the Latin translation of *Newton's* Optics and dedicated "The Doctrine of Chances" to him. Newton returned the compliment by saying to those who questioned him on 'Principia'; "Go to *Mr. De Moivre* : he knows these things better than I do". *De Moivre's* work on "*Annuities upon Lives*" was a popular book in the field of actuarial mathematics.

An interesting fable is often told of *De Moivre's* death. According to the story, *De Moivre* noticed that each day he required a quarter of an hour more sleep than on the preceding day. When this Arithmetic progression reached 24 hours, *De Moivre* passed away.

Answers

Exercise 2(a)

I. 2. (i) -8 (ii) 16 (iii) 256 (iv) i

Exercise 2(b)

I. 1. (i) $2^{1/3}\ cis\left(6k-1\right)\dfrac{\pi}{9},\ k = 0, 1, 2.$

(ii) $cis\left(4k+1\right)\dfrac{\pi}{12},\ k = 0, 1, 2, 3, 4, 5.$

(iii) $2^{1/3}\ cis\left(4k+1\right)\dfrac{\pi}{6},\ k = 0, 1, 2.$

(iv) $2\ cis\left(2k+1\right)\dfrac{\pi}{4},\ k = 0, 1, 2, 3$

(v) $2\ cis\left(2k+1\right)\dfrac{\pi}{5},\ k = 0, 1, 2, 3, 4$

2. -1 3. (i) $2\cos 6\theta$ (ii) $2, 2\omega, 2\omega^2$

6. (i) $3(a^3+b^3)$ (ii) $12ab$ (iii) -8 (iv) 9 (v) -1

(vi) -2 (vii) 32

II. 1. (i) $\pm 1, \pm i$

(ii) $cis\,\frac{(2k+1)\pi}{5}$, $k = 0, 1, 2, 3, 4$

(iii) $\pm 1, \pm i, cis\left(\pm\frac{\pi}{5}\right), cis\left(\pm\frac{3\pi}{5}\right)$

(iv) $cis\,(2k+1)\frac{\pi}{4}$, $k = 0, 1, 2, 3.$

2. $cis\,\frac{\pi}{3}, cis\,\frac{2\pi}{3}, cis\,\frac{4\pi}{3}, cis\,\frac{5\pi}{3}$

3. 5

4. $-1, 1-2\omega, 1-2\omega^2$

5. -4

III. 2. $cis\,\frac{2k\pi}{7}\ (k = 1, 2, 3, 4, 5, 6)$

5. $\frac{1}{2}\left[1 + i\cot\frac{k\pi}{n}\right]$, $k = 1, 2, \ldots, (n-1)$

Chapter 3

Quadratic Expressions

" As great a genius as Archimedes could not invent analytical geometry, for the algebraic knowledge necessary for such an achievement was not available in his time"

- Nathan A. Court

Introduction

Algebra is an important part of mathematics, an understanding of it is essential for the study of most of its advanced branches. With algebra, we can solve problems that would be difficult or impossible to solve with arithmetic alone. Algebra has many practical applications in science, engineering, business and industry.

al-Khwarizmi introduced the solutions of equations. His equations were linear or quadratic. His mathematics was done entirely in words with no symbols. The originality of the concepts and the style of al-Khwarizmi's Algebra are indeed remarkable. The word algorithm is derived from his name.

With a system of algebraic symbols, mathematicians could think in terms of types of problems rather than individual ones. The principal instrument for solving problems in algebra

al-Khwarizmi

(780 - 850)

Muhammad ibn Musa al-Khwarizmi *was a Persian mathematician, astronomer, astrologer and geographer. He was born around 780 A.D. in Khwarizm (now Khiva, Uzbekistan) and died around 850. He worked most of his life as a scholar in the House of Wisdom in Baghdad. His 'Algebra' was the first book on the systematic solutions of linear and quadratic equations.*

is the equation. Algebra is also concerned with expressions and inequalities. In algebra, we learn more about the properties of numbers and about the rules that govern operations with numbers.

In mathematics, we cultivate appreciation of abstract concepts, for which algebra forms the basis. It is the study of properties of abstract mathematical systems.

Now, in this chapter, we discuss some basic concepts of quadratic expressions and equations in one variable, extreme values, changes in sign and magnitude and quadratic inequations.

3.1 Quadratic Expressions, Equations in one Variable

In the present section we discuss about the quadratic equations and their roots.

3.1.1 Definition

A polynomial of the form ax^2+bx+c, *where a, b, c are real or complex numbers and* $a \neq 0$, *is called a quadratic expression in the variable x.*

ax^2+bx+c *is called the standard form of the quadratic expression. In this expression, a is the coefficient of* x^2, *b is the coefficient of x and c is the constant term. The first term* ax^2 *and the second term bx are called the quadratic term and linear term respectively.*

A complex number α *is said to be a zero of the quadratic expression* ax^2+bx+c, *if* $a\alpha^2+b\alpha+c=0$.

3.1.2 Example : $2x^2+5x+7$ and $3ix^2-(i+2)x-5$ are quadratic expressions. $0.x^2+2x+3$ is not a quadratic expression, because the coefficient of x^2 is zero. It is a first degree expression.

3.1.3 Definition

Any equation of the form $ax^2+bx+c=0$, *where a, b, c are real or complex numbers and* $a \neq 0$ *is called a quadratic equation in the variable x. The numbers a, b, c are called the coefficients of this equation.*

3.1.4 Examples

(i) $x^2+2x+3=0$ is a quadratic equation in x.

(ii) $5x^2-8x=2x+4$ is also a quadratic equation, since it can be rewritten as $5x^2-10x-4=0$.

(iii) $2x^3+3x+2=0$ is not a quadratic equation.

(iv) $0.x^2+2x+5=0$ is not a quadratic equation, because the coefficient of x^2 is zero.

(v) $ax^2+c=0$ is a quadratic equation, if $a \neq 0$.

3.1.5 The roots of a quadratic equation

A complex number α is said to be a **root** or **solution** of the quadratic equation $ax^2+bx+c=0$, if $a\alpha^2+b\alpha+c=0$. For example, 2 is a root of the quadratic equation $3x^2-5x-2=0$, since $3.2^2-5.2-2=12-10-2=0$.

3.1.6 Note

The zeros of the quadratic expression ax^2+bx+c are the same as the roots of the quadratic equation $ax^2+bx+c=0$.

3.1.7 Theorem : *The roots of the quadratic equation $ax^2+bx+c=0$ are*

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Proof : α is a root of the quadratic equation $ax^2+bx+c=0$

$$\Leftrightarrow\ a\alpha^2+b\alpha+c=0$$

$$\Leftrightarrow\ 4a[a\alpha^2+b\alpha+c]=0 \qquad (\because a\neq 0)$$

$$\Leftrightarrow\ (2a\alpha+b)^2-b^2+4ac=0$$

$$\Leftrightarrow\ (2a\alpha+b)^2=b^2-4ac$$

$$\Leftrightarrow\ 2a\alpha+b=\pm\sqrt{b^2-4ac}$$

$$\Leftrightarrow\ \alpha=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad (\because a\neq 0).$$

3.1.8 Corollary

A quadratic equation $ax^2+bx+c=0$ has two roots (not necessarily distinct).

3.1.9 Example : We find the roots of the equation $x^2-7x+12=0$

On comparing the given equation with $ax^2+bx+c=0$, we have $a=1,\ b=-7$ and $c=12$.

By Theorem 3.1.7, the roots of $ax^2+bx+c=0$ are

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Therefore, the roots of the given equation are

$$\frac{-(-7)\pm\sqrt{(-7)^2-4(1)(12)}}{2(1)}$$

$$=\frac{7\pm\sqrt{49-48}}{2}=\frac{7\pm1}{2}.$$

Hence the roots of $x^2-7x+12=0$ are 4 and 3.

3.1.10 Definition

$b^2 - 4ac$ is called the discriminant of the quadratic expression $ax^2 + bx + c$ as well as the quadratic equation $ax^2 + bx + c = 0$ and is denoted by the symbol Δ.

3.1.11 Nature of the roots of a quadratic equation

Let α, β be the roots of the quadratic equation $ax^2 + bx + c = 0$, where a, b, c are real numbers. Then the following cases arise

Case (i) $\Delta = 0 \Leftrightarrow \alpha = \beta = -\dfrac{b}{2a}$ (a repeated root or double root of $ax^2 + bx + c = 0$).

Case (ii) $\Delta > 0 \Leftrightarrow \alpha$ and β are real and distinct.

Case (iii) $\Delta < 0 \Leftrightarrow \alpha$ and β are non real complex numbers conjugate to each other.

3.1.12 Note

Let a, b and c be rational numbers, α and β be the roots of the equation $ax^2 + bx + c = 0$. Then

(i) α, β are equal rational numbers if $\Delta = 0$.

(ii) α, β are distinct rational numbers if Δ is the square of a non zero rational number.

(iii) α, β are conjugate surds if $\Delta > 0$ and Δ is not the square of a rational number.

3.1.13 Example : We show that the equation $2x^2 - 6x + 7 = 0$ has no real root.

Here $a = 2,\ b = -6,\ c = 7$

So, $\Delta = b^2 - 4ac = (-6)^2 - 4(2)(7) = -20$.

Therefore the solutions are given by $x = \dfrac{-(-6) \pm \sqrt{-20}}{2(2)} = \dfrac{6 \pm i\sqrt{20}}{4}$

which are non real complex numbers. Hence the given equation has no real root.

3.1.14 Example : Find all k such that the equation $x^2 + 2(k+2)x + 9k = 0$ has equal roots.

Here $a = 1, b = 2(k+2)$ and $c = 9k$.

The condition for the quadratic equation to have equal roots is $b^2 - 4ac = 0$.

i.e., $[2(k+2)]^2 - 4(1)(9k) = 0$

i.e., $(k+2)^2 - 9k = 0$

i.e., $k^2 - 5k + 4 = 0$.

This is a quadratic equation in k.

By Theorem 3.1.7 the roots of this equation are

$$k = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(4)}}{2(1)}.$$

Hence $k = 1$; $k = 4$.

3.1.15 Example : We show that the roots of the equation $x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$ are rational, given that p, q, r are rational.

$$\text{Here } \Delta = (-2p)^2 - 4(1)(p^2 - q^2 + 2qr - r^2)$$

$$= 4(q^2 - 2qr + r^2) = 4(q-r)^2$$

$$= [2(q-r)]^2 .$$

The coefficients of the given equation are rational numbers, since p, q, r are rational numbers.

Since Δ is the square of the rational number $2(q-r)$, the roots of the given equation are rational.

3.1.16 Relation between coefficients and roots of a quadratic equation

1. Let α and β be the roots of the quadratic equation $ax^2 + bx + c = 0$. Then

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

we have $$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= -\frac{2b}{2a} = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

and $$\alpha\beta = \frac{\left(-b + \sqrt{b^2 - 4ac}\right)\left(-b - \sqrt{b^2 - 4ac}\right)}{4a^2}$$

$$= \frac{(-b)^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}, \quad (\text{Note that } a \neq 0)$$

so that $ax^2+bx+c = a(x^2+\frac{b}{a}x+\frac{c}{a})$

$$= a\left[x^2-(\alpha+\beta)x+\alpha\beta\right]$$

$$= a(x-\alpha)(x-\beta).$$

2. If the coefficient of x^2 in a quadratic equation is 1, then

(i) the sum of the roots of the equation is equal to the coefficient of x with its sign changed, and

(ii) the product of the roots is equal to the constant term.

3. If α, β are the roots of $ax^2+bx+c=0$, then the equation can be written as $a(x^2-(\alpha+\beta)x+\alpha\beta)=0$.

i.e., $a\ (x-\alpha)(x-\beta)=0$ i.e., $(x-\alpha)(x-\beta)=0$ or $x^2-(\alpha+\beta)x+\alpha\beta=0$ $(\because a\neq 0)$

3.1.17 Example : Form a quadratic equation whose roots are 3 and – 2.

Since 3 and –2 are roots, the quadratic equation is $x^2-[3+(-2)]x+3(-2)=0$.

$$\text{i.e., } x^2-x-6=0.$$

This is a quadratic equation whose roots are 3 and– 2.

3.1.18 Example : We form a quadratic equation whose roots are $2+\sqrt{3}$ and $2-\sqrt{3}$.

Let $\alpha = 2+\sqrt{3}$ and $\beta = 2-\sqrt{3}$.

Now, $\alpha+\beta = (2+\sqrt{3})+(2-\sqrt{3})=4$,

$\alpha\beta = (2+\sqrt{3})(2-\sqrt{3})=4-2\sqrt{3}+2\sqrt{3}-3=1$.

Since a quadratic equation having roots α and β is of the form

$$x^2-(\alpha+\beta)x+\alpha\beta=0,$$

$x^2-4x+1=0$ is a required equation.

3.1.19 Example : We find a quadratic equation whose roots are $-a+ib$ and $-a-ib$.

Let $\alpha = -a+ib$ and $\beta = -a-ib$. Then $\alpha+\beta = (-a+ib)+(-a-ib)=-2a$

and $\alpha\beta = (-a+ib)(-a-ib) = (-a)^2-(ib)^2 = a^2+b^2$.

So $x^2+2ax+a^2+b^2 = 0$ is a required equation.

3.1.20 Example : If α and β are the roots of the equation $ax^2+bx+c=0$, then form a quadratic equation whose roots are $p\alpha$ and $p\beta$ (p is a real number) in terms of a, b, c, p.

Since α and β are the roots of $ax^2+bx+c=0$, we know that

$$\alpha+\beta = \frac{-b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

A quadratic equation whose roots are $p\alpha$ and $p\beta$ is

$$x^2-(p\alpha+p\beta)x+(p\alpha)(p\beta)=0$$

i.e., $x^2-p(\alpha+\beta)x+p^2\alpha\beta=0$

i.e., $x^2-p\left(\frac{-b}{a}\right)x+p^2\left(\frac{c}{a}\right)=0$

i.e., $ax^2+pbx+p^2c = 0$.

3.1.21 Note 1 : The roots of the quadratic equations

$$a_1x^2+b_1x+c_1=0 \text{ and } a_2x^2+b_2x+c_2=0$$

are identical if and only if (a_1, b_1, c_1) and (a_2, b_2, c_2) are proportional and in this case,

$$a_1x^2+b_1x+c_1=\frac{a_1}{a_2}(a_2x^2+b_2x+c_2).$$

Suppose that the roots of the two equations are identical. Let them be α and β.

Then $\alpha+\beta=\frac{-b_1}{a_1}=\frac{-b_2}{a_2}$ and $\alpha\beta=\frac{c_1}{a_1}=\frac{c_2}{a_2}$.

$$\Rightarrow \frac{b_1}{a_1}=\frac{b_2}{a_2} \text{ and } \frac{c_1}{a_1}=\frac{c_2}{a_2} \qquad \ldots (1)$$

Now $a_1x^2+b_1x+c_1$

$$= a_1\left(x^2+\frac{b_1}{a_1}x+\frac{c_1}{a_1}\right) \left(\because a_1\neq 0\right)$$

$$= a_1\left(x^2+\frac{b_2}{a_2}x+\frac{c_2}{a_2}\right) \text{(from (1))}$$

$$= \frac{a_1}{a_2}\left(a_2x^2+b_2x+c_2\right)$$

$$\therefore\ a_1x^2+b_1x+c_1=\frac{a_1}{a_2}(a_2x^2+b_2x+c_2).$$

Converse is easy.

2. From Note 1 it is clear that if α, β are the roots of $ax^2 + bx + c = 0$ then α, β are the roots of $kax^2 + kbx + kc = 0$ for every real number $k \neq 0$.

3.1.22 Solved Problems

1. Problem : *Find the roots of the equation* $3x^2 + 2x - 5 = 0$.

Solution : By Theorem 3.1.7, the roots of the quadratic equation

$$ax^2 + bx + c = 0 \text{ are } \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Here $a = 3$, $b = 2$ and $c = -5$.

Therefore the roots of the given equation are

$$\frac{-2 \pm \sqrt{(2)^2 - 4(3)(-5)}}{2(3)} = \frac{-2 \pm \sqrt{4 + 60}}{6}.$$

$$= \frac{-2 \pm 8}{6} = 1, \ -\frac{5}{3}.$$

Hence 1 and $-\frac{5}{3}$ are the roots of the given equation.

Another method

We can also obtain these roots in the following way.

$$3x^2 + 2x - 5 = 3x^2 + 5x - 3x - 5$$

$$= x(3x + 5) - 1\,(3x + 5)$$

$$= (x - 1)\,(3x + 5)$$

$$= 3(x - 1)\,(x + \frac{5}{3}).$$

Since 1 and $-\frac{5}{3}$ are the zeros of $3x^2 + 2x - 5$, they are the roots of $3x^2 + 2x - 5 = 0$.

2. Problem : *Find the roots of the equation* $4x^2 - 4x + 17 = 3x^2 - 10x - 17$.

Solution : Given equation can be rewritten as $x^2 + 6x + 34 = 0$. By Theorem 3.1.7, the roots of the quadratic equation

$ax^2 + bx + c = 0$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Here $a = 1$, $b = 6$ and $c = 34$.

Therefore the roots of the given equation are

$$\frac{-6 \pm \sqrt{(6)^2 - 4(1)(34)}}{2(1)} = \frac{-6 \pm \sqrt{-100}}{2}$$

$$= \frac{-6 \pm 10i}{2} \quad (\text{since } i^2 = -1)$$

$$= -3 + 5i,\ -3 - 5i.$$

Hence the roots of the given equation are $-3 + 5i$ and $-3 - 5i$.

3. Problem : *Find the roots of the equation* $\sqrt{3}x^2 + 10x - 8\sqrt{3} = 0$.

Solution : The roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Here $a = \sqrt{3}, b = 10,$ and $c = -8\sqrt{3}$.

Therefore the roots of the given equation are

$$\frac{-10 \pm \sqrt{(10)^2 - 4(\sqrt{3})(-8\sqrt{3})}}{2\sqrt{3}}$$

$$= \frac{-10 \pm 14}{2\sqrt{3}} = \frac{4}{2\sqrt{3}}, -\frac{24}{2\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}, -4\sqrt{3}.$$

Hence the roots of the given equation are $\dfrac{2}{\sqrt{3}}$ and $-4\sqrt{3}$.

4. Problem : *Find the nature of the roots of the equation* $4x^2 - 20x + 25 = 0$

Solution : Here $a = 4, b = -20$ and $c = 25$.

Hence $\Delta = b^2 - 4ac = (-20)^2 - 4(4)(25) = 0.$

Since Δ is zero and a, b, c are real, the roots of the given equation are real and equal.

5. Problem : *Find the nature of the roots of the equation* $3x^2+7x+2=0$.

Solution : Here $a=3,\ b=7$ and $c=2$.

Hence $\Delta = b^2-4ac = (7)^2-4(3)(2) = 25 = 5^2 > 0$.

Since $\Delta = 5^2$ is a square number, the roots of the given equation are rational and unequal.

6. Problem : *Find the values of m for which the equation* $x^2-2(1+3m)x+7(3+2m)=0$ *will have equal roots.*

Solution : The given equation will have equal roots iff its discriminant is 0.

$$\begin{aligned}\text{Here } \Delta &= \{-2(1+3m)\}^2-4(1)7(3+2m)\\ &= 4(1+3m)^2-28(3+2m) = 4(9m^2-8m-20)\\ &= 4(m-2)(9m+10) = 36(m-2)\left(m+\frac{10}{9}\right)\end{aligned}$$

Hence $\Delta = 0 \Leftrightarrow m = 2$ or $m = -\dfrac{10}{9}$.

$\therefore$ The given equation will have equal roots for $m = 2$ or $-\dfrac{10}{9}$

7. Problem : *If* α and β *are the roots of the equation* $ax^2+bx+c=0$, find the values of $\alpha^2+\beta^2$ and $\alpha^3+\beta^3$ in terms of *a, b, c.*

Solution : From the hypothesis

$$\alpha+\beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

$$\begin{aligned}\therefore\ \alpha^2+\beta^2 &= (\alpha+\beta)^2-2(\alpha\beta)\\ &= \left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right) = \frac{b^2}{a^2}-\frac{2ac}{a^2} = \frac{b^2-2ac}{a^2}\end{aligned}$$

$$\begin{aligned}\text{and } \alpha^3+\beta^3 &= (\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)\\ &= (\alpha+\beta)\left[(\alpha+\beta)^2-2\alpha\beta-\alpha\beta\right]\\ &= (\alpha+\beta)\left[(\alpha+\beta)^2-3(\alpha\beta)\right]\\ &= \left(-\frac{b}{a}\right)\left[\left(\frac{-b}{a}\right)^2-3\left(\frac{c}{a}\right)\right]\end{aligned}$$

$$= -\frac{b}{a}\left(\frac{b^2}{a^2}-\frac{3c}{a}\right)=\frac{3abc-b^3}{a^3}.$$

8. Problem : *Form a quadratic equation whose roots are* $2\sqrt{3}-5$ and $-2\sqrt{3}-5$.

Solution : Let $\alpha = 2\sqrt{3}-5$ and $\beta=-2\sqrt{3}-5$.

Then $\alpha+\beta = (2\sqrt{3}-5)+(-2\sqrt{3}-5)=-10$

and $\alpha\beta = (2\sqrt{3}-5)(-2\sqrt{3}-5)$

$= (2\sqrt{3})(-2\sqrt{3})-(2\sqrt{3})(5)-5(-2\sqrt{3})-5(-5)$

$= -12-10\sqrt{3}+10\sqrt{3}+25=13$.

Therefore $x^2-(\alpha+\beta)x+\alpha\beta=0$ becomes

$x^2-(-10)x+13=0$

i.e., $x^2+10x+13=0$.

This is a quadratic equation whose roots are $2\sqrt{3}-5$ and $-2\sqrt{3}-5$.

9. Problem : *Let* α and β *be the roots of the quadratic equation* $ax^2+bx+c=0$. *If* $c\neq 0$, *then form the quadratic equation whose roots are* $\frac{1-\alpha}{\alpha}$ and $\frac{1-\beta}{\beta}$.

Solution : From the hypothesis we have

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

Since $c\neq 0$, $\alpha\neq 0$ and $\beta\neq 0$

$$\frac{1-\alpha}{\alpha}+\frac{1-\beta}{\beta} = \frac{\beta(1-\alpha)+\alpha(1-\beta)}{\alpha\beta}=\frac{\alpha+\beta-2\alpha\beta}{\alpha\beta}$$

$$= \frac{\alpha+\beta}{\alpha\beta}-2=\frac{\left(-\frac{b}{a}\right)}{\left(\frac{c}{a}\right)}-2$$

$$= -\frac{b}{c}-2=-\left(2+\frac{b}{c}\right)$$

and $$\left(\frac{1-\alpha}{a}\right)\left(\frac{1-\beta}{\beta}\right) = \frac{1-(\alpha+\beta)+\alpha\beta}{\alpha\beta}$$

$$= \frac{1-\left(-\frac{b}{a}\right)+\left(\frac{c}{a}\right)}{\left(\frac{c}{a}\right)}$$

$$= \frac{a+b+c}{c}.$$

Therefore $x^2 - \left\{-\left(2+\frac{b}{c}\right)\right\}x + \frac{a+b+c}{c} = 0$ is the required equation.

10. Problem : *Find a quadratic equation, the sum of whose roots is 1 and the sum of the squares of the roots is 13.*

Solution : Let α and β be the roots of a required equation.

Then $\alpha+\beta = 1$ and $\alpha^2+\beta^2 = 13.$

Since $\alpha\beta = \frac{1}{2}\left[(\alpha+\beta)^2-(\alpha^2+\beta^2)\right],$

$$\alpha\beta = \frac{1}{2}\left[(1)^2-13\right] = -6.$$

Therefore $x^2-(\alpha+\beta)x+\alpha\beta=0$ becomes $x^2-x-6=0$. This is a required equation.

Equations reducible to quadratic equations : We now explain by some illustrations how to solve some equations which are reducible to quadratic equations by suitable substitutions.

11. Problem : *Solve* $x^{\frac{2}{3}}+x^{\frac{1}{3}}-2=0$.

Solution : On taking $x^{\frac{1}{3}}=t$, the given equation becomes $t^2+t-2=0$, which is a quadratic equation in *t*.

Hence a complex number α is a solution of the equation

$x^{\frac{2}{3}}+x^{\frac{1}{3}}-2=0$, if there exists λ such that $\lambda^2+\lambda-2=0$ and $\lambda^3=\alpha$.

Therefore the set of all solutions of the given equation is

$$\left\{t^3 : t\in \mathbf{C} \text{ and } t^2+t-2=0\right\}.$$

Since $t^2+t-2 = (t-1)(t+2)$,

$$t^2+t-2=0 \Leftrightarrow t\in\{-2,1\}.$$

$$x^{\frac{1}{3}} = t \Rightarrow x = t^3.$$

Hence $\{t^3 : t \in \mathbf{C} \text{ and } t^2 + t - 2 = 0\} = \{(-2)^3, 1^3\} = \{-8, 1\}.$

Therefore the solution set of the given equation is $\{-8, 1\}$.

12. Problem : *Solve* $7^{1+x} + 7^{1-x} = 50$ for real x.

Solution : The given equation can be written as

$$7.7^x + \frac{7}{7^x} - 50 = 0$$

i.e., $\quad 7.7^x.7^x + 7 - 50.7^x = 0$

i.e., $\quad 7.7^{2x} - 50.7^x + 7 = 0$.

On taking $7^x = t$, this equation becomes

$$7t^2 - 50t + 7 = 0$$

i.e., $\quad 7t^2 - 49t - t + 7 = 0$

i.e., $\quad 7t(t-7) - (t-7) = 0$

i.e., $\quad (7t-1)(t-7) = 0$.

$\frac{1}{7}$ and 7 are the roots of the above equation.

i.e., $\quad 7^x = \frac{1}{7}$ or $7^x = 7$.

$$\Leftrightarrow 7^x = 7^{-1} \text{ or } 7^x = 7^1$$

$$\Leftrightarrow x = -1 \text{ or } x = 1$$

When x is real, $7^x = 7^{-1}$ or $7^x = 7^1 \Rightarrow x = -1$ or $x = 1$.

Therefore the solution set of the given equation is $\{-1, 1\}$.

13. Problem : *Solve* $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$.

Solution : On taking $\sqrt{\frac{x}{1-x}} = t$, the given equation becomes

$$t + \frac{1}{t} = \frac{13}{6}$$

i.e., $\dfrac{t^2+1}{t}=\dfrac{13}{6}$

i.e., $6t^2-13t+6=0$

i.e., $6t^2-9t-4t+6=0$

i.e., $3t(2t-3)-2(2t-3)=0$

i.e., $(2t-3)(3t-2)=0$.

$\dfrac{3}{2}$ and $\dfrac{2}{3}$ are the roots of the above equation.

Hence $\sqrt{\dfrac{x}{1-x}}+\sqrt{\dfrac{1-x}{x}}=\dfrac{13}{6} \Leftrightarrow \sqrt{\dfrac{x}{1-x}}=\dfrac{3}{2}$ or $\sqrt{\dfrac{x}{1-x}}=\dfrac{2}{3}$

$$\Leftrightarrow \frac{x}{1-x}=\frac{9}{4} \text{ or } \frac{x}{1-x}=\frac{4}{9}$$

$$\Leftrightarrow 9-9x=4x \text{ or } 4-4x=9x$$

$$\Leftrightarrow 13x=9 \text{ or } 13x=4$$

$$\Leftrightarrow x=\frac{9}{13} \text{ or } x=\frac{4}{13}.$$

It can be verified that $x=\dfrac{9}{13}$ satisfies equation $\sqrt{\dfrac{x}{1-x}}=\dfrac{3}{2}$

and $x=\dfrac{4}{13}$ satisfies equation $\sqrt{\dfrac{x}{1-x}}=\dfrac{2}{3}$

Therefore the solution set of the given equation is $\left\{\dfrac{9}{13},\dfrac{4}{13}\right\}$.

14. Problem : *Find all numbers which exceed their square root by 12.*

Solution : Let x be any such number.

Then $x=\sqrt{x}+12$ i.e., $x-12=\sqrt{x}$ ... (1)

On squaring both sides and simplifying we obtain

$$(x-12)^2=x$$

i.e., $x^2 - 24x + 144 = x$

i.e., $x^2 - 25x + 144 = 0$

i.e., $x(x-16) - 9(x-16) = 0$

i.e., $(x-16)(x-9) = 0.$

The roots of the equation $(x-16)(x-9) = 0$ are 9 and 16.

But $x = 9$ does not satisfy equation (1), while $x = 16$ satisfies (1). Therefore the required number is 16.

15. Problem : *Prove that there is a unique pair of consecutive positive odd integers such that the sum of their squares is 290 and find it.*

Solution : Since two consecutive odd integers differ by 2, we have to prove that there is a unique positive odd integer x such that

$$x^2 + (x+2)^2 = 290 \qquad \text{... (1)}$$

$$\begin{aligned} x^2 + (x+2)^2 = 290 &\Leftrightarrow x^2 + x^2 + 4x + 4 = 290 \\ &\Leftrightarrow 2x^2 + 4x - 286 = 0 \\ &\Leftrightarrow x^2 + 2x - 143 = 0 \\ &\Leftrightarrow x^2 + 13x - 11x - 143 = 0 \\ &\Leftrightarrow x(x+13) - 11(x+13) = 0 \\ &\Leftrightarrow (x+13)\ (x-11) = 0 \\ &\Leftrightarrow x \in \{-13, 11\}. \end{aligned}$$

Hence 11 is the only positive odd integer satisfying equation (1).

Therefore (11, 13) is the unique pair of integers which satisfies the given condition.

16. Problem : *The cost of a piece of cable wire is Rs. 35/-. If the length of the piece of wire is 4 meters more and each meter costs Rs. 1/- less, the cost would remain unchanged. What is the length of the wire?*

Solution : Let the length of the piece of wire be 'l' meters and the cost of each meter be Rs. x/-.

By the given conditions $lx = 35$ ... (1)

Also, $(l+4)(x-1) = 35$

i.e., $lx - l + 4x - 4 = 35$ ... (2)

From (1) and (2), $35 - l + 4x - 4 = 35$

i.e., $4x = l + 4$

Therefore $x = \dfrac{l+4}{4}.$

On substituting this value of 'x' in (1) and simplifying, we get

$$l\left(\frac{l+4}{4}\right)=35$$

i.e., $l^2+4l-140=0$

i.e., $l^2+14l-10l-140=0$

i.e., $l(l+14)-10(l+14)=0$

i.e., $(l+14)(l-10)=0$.

The roots of the equation $(l+14)(l-10)=0$ are -14 and 10.

Since the length can not be negative, $l=10$.

Therefore the length of the piece of wire is 10 meters.

17. Problem : *One fourth of a herd of goats was seen in the forest. Twice the square root of the number in the herd had gone up the hill and the remaining 15 goats were on the bank of the river. Find the total number of goats.*

Solution : Let the number of goats in the herd be 'x'.

By the given conditions, the number of goats seen in the forest is $\frac{x}{4}$, the number of goats gone up the hill is $2\sqrt{x}$ and the number of the remaining goats which were on the bank of the river is 15.

Therefore $\frac{x}{4}+2\sqrt{x}+15=x$

i.e., $x+8\sqrt{x}+60=4x$

i.e., $3x-8\sqrt{x}-60=0$.

On taking $\sqrt{x}=t$, this equation becomes

$$3t^2-8t-60=0$$

i.e., $3t^2-18t+10t-60=0$

i.e., $3t(t-6)+10(t-6)=0$

i.e., $(t-6)(3t+10)=0$.

The roots of the equation $(t-6)(3t+10)=0$ are 6 and $-\frac{10}{3}$.

Hence $3x-8\sqrt{x}-60=0 \Leftrightarrow \sqrt{x}=6$ (since $\sqrt{x}$ is non negative) $\Leftrightarrow x=36$.

Therefore the total number of goats in the herd is 36.

18. Problem : *In a cricket match Anil took one wicket less than twice the number of wickets taken by Ravi. If the product of the number of wickets taken by them is 15, find the number of wickets taken by each of them.*

Solution : Let the number of wickets taken by Anil and Ravi be x and y respectively.

Then $x = 2y - 1$ ... (1)

$xy = 15$ ... (2)

From (1) and (2), $(2y-1)y = 15$

i.e., $2y^2 - y - 15 = 0$

i.e., $2y^2 - 6y + 5y - 15 = 0$

i.e., $2y(y-3) + 5(y-3) = 0$

i.e., $(y-3)(2y+5) = 0$.

The roots of the equation $(y-3)(2y+5) = 0$ are 3 and $-\frac{5}{2}$.

Since the number of wickets must be positive integer $y = 3$.

From (2) we get $3x = 15$, i.e., $x = 5$.

Therefore the wickets taken by Anil and Ravi are 5 and 3 respectively.

19. Problem : *Some points on a plane are marked such that no three of them are collinear and then they are connected pairwise by line segments. If the total number of line segments formed is 10, find the number of marked points on the plane.*

Solution : Let the number of points marked on the plane be 'x'. Since each point is joined to the remaining $(x-1)$ points, the number of line segments having a given point as an end point is $(x-1)$. Hence the total number of line segments formed appears to be $x(x-1)$. But in this counting, each line segment is counted exactly twice at each of its end points. Hence the total number of line segments actually formed is $\frac{x(x-1)}{2}$.

Therefore by hypothesis $\frac{x(x-1)}{2} = 10$.

i.e., $x^2 - x - 20 = 0$

i.e., $(x-5)(x+4) = 0$.

The roots of the equation $(x-5)(x+4) = 0$ are -4 and 5 .

x can not be negative, so $x = 5$. Therefore the number of points marked on the plane is 5.

Exercise 3(a)

I. 1. Find the roots of the following equations.

(i) $x^2 - 7x + 12 = 0$ (ii) $-x^2 + x + 2 = 0$

(iii) $2x^2 + 3x + 2 = 0$ (iv) $\sqrt{3}x^2 + 10x - 8\sqrt{3} = 0$

(v) $6\sqrt{5}x^2 - 9x - 3\sqrt{5} = 0$

2. Form quadratic equations whose roots are :

(i) $2, 5$ (ii) $\frac{m}{n}, -\frac{n}{m}$ $(m \neq 0, n \neq 0)$

(iii) $\frac{p-q}{p+q}, -\frac{p+q}{p-q}$ $(p \neq \pm q)$ (iv) $7 \pm 2\sqrt{5}$ (v) $-3 \pm 5i$

3. Find the nature of the roots of the following equations without finding the roots.

(i) $2x^2 - 8x + 3 = 0$ (ii) $9x^2 - 30x + 25 = 0$

(iii) $x^2 - 12x + 32 = 0$ (iv) $2x^2 - 7x + 10 = 0$

4. If α, β are the roots of the equation $ax^2 + bx + c = 0$, find the values of the following expressions in terms of a, b, c.

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$ (ii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

(iii) $\alpha^4\beta^7 + \alpha^7\beta^4$ (iv) $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$, if $c \neq 0$

(v) $\frac{\alpha^2 + \beta^2}{\alpha^{-2} + \beta^{-2}}$, if $c \neq 0$

5. Find the values of m for which the following equations have equal roots?

(i) $x^2 - 15 - m(2x - 8) = 0$

(ii) $(m+1)x^2 + 2(m+3)x + (m+8) = 0$

(iii) $x^2 + (m+3)x + (m+6) = 0$

(iv) $(3m+1)x^2 + 2(m+1)x + m = 0$

(v) $(2m+1)x^2 + 2(m+3)x + (m+5) = 0$

6. If α and β are the roots of $x^2 + px + q = 0$, form a quadratic equation whose roots are $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$.

7. If $x^2+bx+c=0, x^2+cx+b=0 (b\neq c)$ have a common root, then show that $b+c+1=0$.

8. Prove that the roots of $(x-a)(x-b)=h^2$ are always real.

9. Find the condition that one root of the quadratic equation $ax^2+bx+c=0$ shall be n times the other, where n is a positive integer.

10. Find two consecutive positive even integers, the sum of whose squares is 340.

II. 1. If x_1, x_2 are the roots of the quadratic equation $ax^2+bx+c=0$ and $c\neq 0$, find the value of $(ax_1+b)^{-2}+(ax_2+b)^{-2}$ in terms of a, b, c.

2. If α, β are the roots of the quadratic equation $ax^2+bx+c=0$, find a quadratic equation whose roots are $\alpha^2+\beta^2$ and $\alpha^{-2}+\beta^{-2}$.

Solve the following equations :

3. $2x^4+x^3-11x^2+x+2=0$

4. $3^{1+x}+3^{1-x}=10$

5. $4^{x-1}-3.2^{x-1}+2=0$

6. $\sqrt{\frac{x}{x-3}}+\sqrt{\frac{x-3}{x}}=\frac{5}{2}$, when $x\neq 0$ and $x\neq 3$

7. $\sqrt{\frac{3x}{x+1}}+\sqrt{\frac{x+1}{3x}}=2$, when $x\neq 0$ and $x\neq -1$

8. $2\left(x+\frac{1}{x}\right)^2-7\left(x+\frac{1}{x}\right)+5=0$, when $x\neq 0$

9. $\left(x^2+\frac{1}{x^2}\right)-5\left(x+\frac{1}{x}\right)+6=0$, when $x\neq 0$

10. Find a quadratic equation for which the sum of the roots is 7 and the sum of the squares of the roots is 25.

Now we shall discuss a theorem which gives the condition for a pair of quadratic equations to have a common root.

3.1.23 Theorem : *A necessary and sufficient condition for the quadratic equations* $a_1x^2+b_1x+c_1=0$ *and* $a_2x^2+b_2x+c_2=0$ *to have a common root is*

$$(c_1a_2-c_2a_1)^2=(a_1b_2-a_2b_1)(b_1c_2-b_2c_1).$$

Proof : Necessity

Let α be a common root of the given equations.

Then $a_1\alpha^2+b_1\alpha+c_1=0$ ... (1)

$a_2\alpha^2+b_2\alpha+c_2=0$ ... (2)

On multiplying equation (1) by a_2, equation (2) by a_1 and then subtracting the latter from the former, we get

$$a_2b_1\alpha-a_1b_2\alpha+a_2c_1-a_1c_2=0$$

i.e., $\alpha(a_2b_1-a_1b_2)=a_1c_2-a_2c_1$

i.e., $\alpha(a_1b_2-a_2b_1)=c_1a_2-c_2a_1$. ... (3)

On multiplying equation (1) by b_2, equation (2) by b_1 and then subtracting the latter from the former, we get

$\alpha^2(a_1b_2-a_2b_1)=b_1c_2-b_2c_1$ ... (4)

On squaring both sides of equation (3) and using (4) we obtain

$$(b_1c_2-b_2c_1)(a_1b_2-a_2b_1)=(c_1a_2-c_2a_1)^2$$

Sufficiency

Suppose that $(a_1b_2-a_2b_1)(b_1c_2-b_2c_1)=(c_1a_2-c_2a_1)^2$ ... (5)

Case (i) $a_1b_2-a_2b_1=0$

Then $c_1a_2-c_2a_1=0$

So $a_1b_2=a_2b_1$ and $c_1a_2=c_2a_1$.

Hence $\dfrac{b_1}{a_1}=\dfrac{b_2}{a_2}$ and $\dfrac{c_1}{a_1}=\dfrac{c_2}{a_2}$ (since $a_1\neq 0$ and $a_2\neq 0$)

Now, $$a_1x^2 + b_1x + c_1 = a_1\left(x^2 + \frac{b_1}{a_1}x + \frac{c_1}{a_1}\right) = a_1\left(x^2 + \frac{b_2}{a_2}x + \frac{c_2}{a_2}\right)$$

Hence $$a_1x^2 + b_1x + c_1 = \frac{a_1}{a_2}(a_2x^2 + b_2x + c_2).$$

Hence $$a_1x^2 + b_1x + c_1 = 0 \Leftrightarrow a_2x^2 + b_2x + c_2 = 0 \quad \left(\text{since } \frac{a_1}{a_2} \neq 0\right)$$

Therefore the given equations have the same roots.

Case (ii) $a_1b_2 - a_2b_1 \neq 0$

Let $\alpha = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

Then $$a_1\alpha^2 + b_1\alpha + c_1 = a_1\left[\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right]^2 + b_1\left[\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right] + c_1$$

$$= \frac{a_1(b_1c_2 - b_2c_1)}{(a_1b_2 - a_2b_1)} + \frac{b_1(c_1a_2 - c_2a_1)}{(a_1b_2 - a_2b_1)} + c_1 \text{ (by equation (5))}$$

$$= \frac{a_1(b_1c_2 - b_2c_1) + b_1(c_1a_2 - c_2a_1) + c_1(a_1b_2 - a_2b_1)}{a_1b_2 - a_2b_1} = 0.$$

i.e., $a_1\alpha^2 + b_1\alpha + c_1 = 0$.

Similarly we can prove that $a_2\alpha^2 + b_2\alpha + c_2 = 0$.

Thus α is a common root of the given equations.

3.1.24 Example : If $x^2 + 4ax + 3 = 0$ and $2x^2 + 3ax - 9 = 0$ have a common root, then find the values of a and the common roots.

By Theorem 3.1.23, the condition for two quadratic equations

$a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ to have a common root is

$(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)\,(b_1c_2 - b_2c_1)$

Here $a_1 = 1,\ b_1 = 4a,\ c_1 = 3;\quad a_2 = 2,\ b_2 = 3a,\ c_2 = -9.$

Therefore the above condition becomes

$$[(3)(2) - (-9)(1)]^2 = [(1)(3a) - (2)(4a)][(4a)(-9) - (3a)(3)].$$

i.e., $$225 = (-5a)(-45a) = 225a^2$$

i.e., $$a^2 = 1, \text{ so } a = \pm 1.$$

If $a=1$ then the given equations reduce to

$$x^2+4x+3=0\ ;\quad 2x^2+3x-9=0$$

i.e., $(x+1)(x+3)=0\ ;\ (2x-3)(x+3)=0$.

Therefore the roots of the given equations are respectively

$$-3,-1 \text{ and } -3,\frac{3}{2}.$$

In this case the common root of the given equations is -3.

Similarly we can show that when $a=-1$ the common root of the given equations is 3.

Another method : If α is the common root them

$$\alpha^2+4a\alpha+3=0 \qquad \text{...(i)}$$

$$2\alpha^2+3a\alpha-9=0 \qquad \text{...(ii)}$$

Multiplying (i) by 2 and subtracting from (ii)

we get $-5a\,\alpha-15=0$

$$\Rightarrow \quad a\alpha+3=0$$

$$\Rightarrow \quad a\alpha=-3.$$

$$\Rightarrow \quad \alpha=\frac{-3}{a}.$$

substituting in (i) we get $\dfrac{9}{a^2}+4a\left(\dfrac{-3}{a}\right)+3=0$

$$\Rightarrow \frac{9}{a^2}-12+3=0$$

$$\Rightarrow \frac{9}{a^2}-9=0$$

$$\Rightarrow a=\pm 1.$$

$$\Rightarrow \alpha=\pm 3.$$

3.1.25 Some properties of quadratic equations

(i) When $a\neq 0$ and $c\neq 0$, $b=0$ we say that $ax^2+c=0$ is a proper quadratic equation.

If $\alpha,\ \beta$ are the roots of the quadratic equation $ax^2+c=0$, then

$$\alpha+\beta=0,\quad \alpha\beta=\frac{c}{a} \quad \text{and} \quad \Delta=-4ac\neq 0.$$

In this case the roots have the same absolute value.

(ii) If α, β are the roots of $ax^2 + bx + a = 0$ $(a \neq 0)$ then $\alpha\beta = \frac{a}{a} = 1$, so that neither α nor β is zero and $\alpha = \frac{1}{\beta}$ and $\beta = \frac{1}{\alpha}$. In this case each root is the reciprocal of the other.

(iii) If the signs of a $(\neq 0)$ and c $(\neq 0)$ in the quadratic equation $ax^2 + bx + c = 0$ are the same, the product of the roots $\frac{c}{a}$ is positive, and hence if the roots are real, they have the same sign. On the other hand, if a and c have different signs, the product of the roots $\frac{c}{a}$ is negative, and hence if the roots are real, they have different signs.

3.1.26 Note: If the coefficients and one root of a quadratic equation are real, then the second root is also real.

3.1.27 Note : Let α, β to be roots of the quadratic equation $f(x) = ax^2 + bx + c = 0$.

(i) If $c \neq 0$, $\alpha\beta \neq 0$, then $\frac{1}{\alpha}, \frac{1}{\beta}$ are the roots of the quadratic equation $f\left(\frac{1}{x}\right) = 0.$

(ii) Then $\alpha + k$, $\beta + k$ are roots of the quadratic equation $f(x-k) = 0.$

(iii) Then $-\alpha, -\beta$ are roots of the quadratic equation $f(-x) = 0.$

(iv) If $k \neq 0$, then $k\alpha, k\beta$ are roots of the quadratic equation $f\left(\frac{x}{k}\right) = 0.$

3.2 Sign of quadratic expressions - Change in signs and Maximum and Minimum values

If a, b are non zero real numbers, we say that a and b have the same sign, if both a and b are positive or both of them are negative.

Now we shall discuss how the sign of a quadratic expression $ax^2 + bx + c$ with real coefficients depends on the coefficient a of x^2 and the nature of the roots of the equation $ax^2 + bx + c = 0$.

3.2.1 Theorem : *Let* $a, b, c \in \mathbf{R}$ and $a \neq 0$. *Then the roots of* $ax^2 + bx + c = 0$ *are non-real complex numbers if and only if* $ax^2 + bx + c$ *and* a *have the same sign for all* $x \in \mathbf{R}$.

Proof : The condition for the equation $ax^2 + bx + c = 0$ to have non-real complex roots is $b^2 - 4ac < 0$, i.e., $4ac - b^2 > 0$.

We have $ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$

$$= a\left[x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right].$$

Hence $\dfrac{ax^2 + bx + c}{a} = \left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac - b^2}{4a^2}$. ... (1)

Now clearly $\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac - b^2}{4a^2} \geq \dfrac{4ac - b^2}{4a^2}$ $\forall x \in \mathbf{R}$.

Therefore $\dfrac{ax^2 + bx + c}{a} > 0$ $\forall$ $x \in \mathbf{R}$, if $4ac - b^2 > 0$.

Conversely suppose that $\dfrac{ax^2 + bx + c}{a} > 0$ $\forall x \in \mathbf{R}$, so that by (1), we have

$$\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2} > 0 \ \forall x \in \mathbf{R}.$$

On taking $x = -\dfrac{b}{2a}$ in the above inequality, we obtain $\dfrac{4ac - b^2}{4a^2} > 0$.

Hence $4ac - b^2 > 0$, so that $b^2 - 4ac < 0$.

Thus $b^2 - 4ac < 0$ if and only if $ax^2 + bx + c$ and a have the same sign for all real x.

3.2.2 Theorem : *Let $a, b, c \in \mathbf{R}$ and $a \neq 0$. If the equation $ax^2 + bx + c = 0$ has equal roots, then $ax^2 + bx + c$ and a have the same sign for all real x, except for $x = -\dfrac{b}{2a}$.*

Proof : The condition for the equation $ax^2 + bx + c = 0$ to have equal roots is $b^2 - 4ac = 0$.

In this case $\dfrac{ax^2 + bx + c}{a} = \left(x + \dfrac{b}{2a}\right)^2 > 0$, for $x \neq -\dfrac{b}{2a}$

$$= 0, \text{ for } x = -\frac{b}{2a}$$

Therefore when $b^2 - 4ac = 0$, $ax^2 + bx + c$ and a have the same sign for all real x, except for

$$x = -\frac{b}{2a}.$$

3.2.3 Theorem : *Let* $a, b, c \in \mathbf{R}$ and $a \neq 0$ *such that the equation* $ax^2 + bx + c = 0$ *has real roots* α *and* β *with* $\alpha < \beta$. *Then*

(i) *for* $\alpha < x < \beta, ax^2 + bx + c$ and a *have opposite signs,*

(ii) *for* $x < \alpha$ or $x > \beta$, $ax^2 + bx + c$ and a *have the same sign.*

Proof : Since α, β are the roots of $ax^2 + bx + c = 0$, we have

$$ax^2 + bx + c = a(x-\alpha)(x-\beta)$$

Therefore $\dfrac{ax^2+bx+c}{a} = (x-\alpha)(x-\beta)$ ... (1)

(i) When $\alpha < x < \beta$, we have $x - \alpha > 0$ and $x - \beta < 0$, so that, by (1)

$$\frac{ax^2+bx+c}{a} < 0 .$$

Hence $ax^2 + bx + c$ and a have opposite signs when $\alpha < x < \beta$.

(ii) When $x < \alpha$, $x - \alpha < 0$ and $x - \beta < 0$ (since $\alpha < \beta$), so that, by (1)

$$\frac{ax^2+bx+c}{a} > 0 .$$

When $x > \beta$, $x - \beta > 0$ and $x - \alpha > 0$ (since $\beta > \alpha$), so that, by (1)

$$\frac{ax^2+bx+c}{a} > 0.$$

Thus for $x < \alpha$ or $x > \beta$, $ax^2 + bx + c$ and a have the same sign.

3.2.4 Example : We determine the sign of the following expressions for $x \in \mathbf{R}$

(i) $x^2 + x + 1$ (ii) $x^2 - 5x + 6$

(i) The roots of $x^2 + x + 1 = 0$ are $\dfrac{-1 \pm \sqrt{3}\,i}{2}$, which are non real.

Therefore $x^2 + x + 1$ and the coefficient of x^2 have the same sign.

Hence $x^2 + x + 1 > 0$ for all real x.

Another method : $x^2 + x + 1 = \left(x + \dfrac{1}{2}\right)^2 - \dfrac{1}{4} + 1 = \left(x + \dfrac{1}{2}\right)^2 + \dfrac{3}{4} \geq \dfrac{3}{4} > 0$ for all x.

(ii) The roots of $x^2 - 5x + 6 = 0$ are 2 and 3, which are real.

Therefore if $2 < x < 3$, then $x^2 - 5x + 6$ and the coefficient of x^2 have opposite signs.

Also, if $x < 2$ or $x > 3$, then $x^2 - 5x + 6$ and the coefficient of x^2 have the same sign.

Hence, for the case $2 < x < 3$, $x^2 - 5x + 6 < 0$ and for $x < 2$ or $x > 3$, $x^2 - 5x + 6 > 0$.

Another method: Since $x^2 - 5x + 6 = (x-2)(x-3)$, $x^2 - 5x + 6 > 0 \Leftrightarrow x - 2$ and $x - 3$ have the same sign. This happens when $x < 2$ or $x > 3$.

3.2.5 Maximum and minimum values

The extreme values of a quadratic expression with real coefficients depend on the sign of the coefficient of x^2. We establish this fact in the following theorem.

3.2.6 Theorem : *Suppose that* $a, b, c \in \mathbf{R}$, $a \neq 0$ and $f(x) = ax^2 + bx + c$

(i) *If* $a > 0$, *then* $f(x)$ *has absolute mimumum at* $x = -\dfrac{b}{2a}$ *and the minimum value is* $\dfrac{4ac - b^2}{4a}$.

(ii) *If* $a < 0$, *then* $f(x)$ *has absolute maximum at* $x = -\dfrac{b}{2a}$ *and the maximum value is* $\dfrac{4ac - b^2}{4a}$.

Proof : We know that $f(x) = ax^2 + bx + c = a\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac - b^2}{4a}$ ... (1)

(i) Let $a > 0$. then $f(x) \geq \dfrac{4ac - b^2}{4a}$ $\forall x \in \mathbf{R}$

and when $x = -\dfrac{b}{2a}$, we have $f(x) = \dfrac{4ac - b^2}{4a}$, by (1).

Therefore, for $a > 0$, $f(x)$ has absolute minimum at $x = -\dfrac{b}{2a}$ and the minimum value is $\dfrac{4ac - b^2}{4a}$. This is shown in Fig.3.1. Further it is clear from the figure that in this case $f(x)$ does not have maximum.

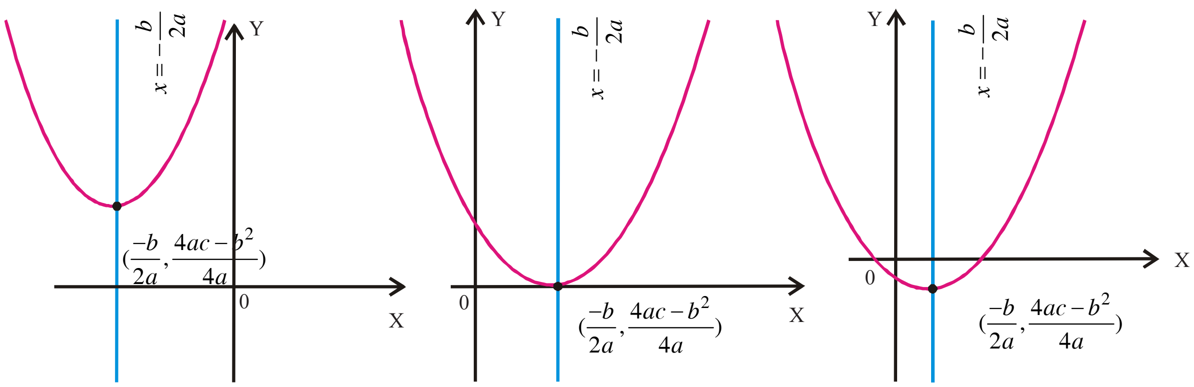


Fig. 3.1

(ii) Let $a < 0$. then $f(x) \le \dfrac{4ac-b^2}{4a}$ $\forall\ x \in \mathbf{R}$

and when $x = -\dfrac{b}{2a}$, we have $f(x) = \dfrac{4ac-b^2}{4a}$, by (1).

Therefore for $a < 0$, $f(x)$ has absolute maximum at $x = -\dfrac{b}{2a}$ and the maximum value is $\dfrac{4ac-b^2}{4a}$.

This is shown in Fig.3.2. Further it is clear that in this case $f(x)$ does not have minimum.

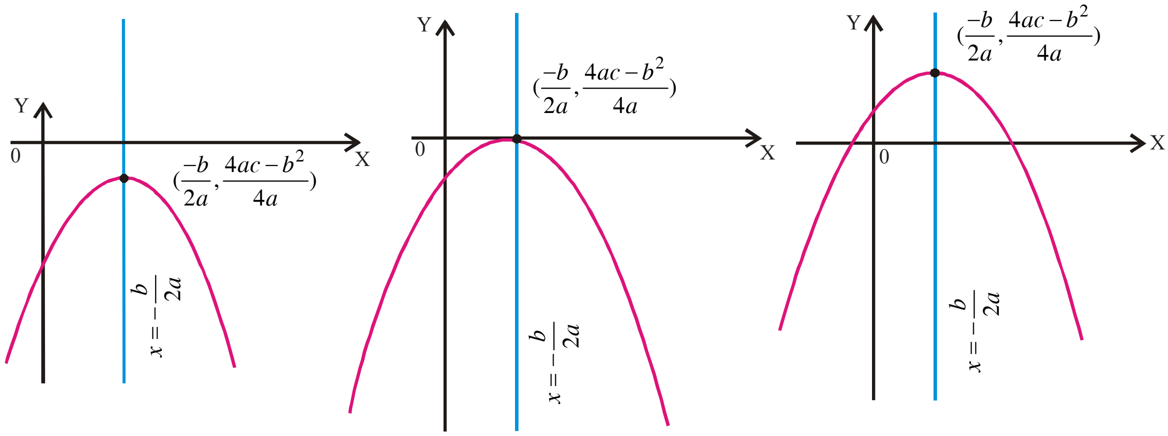


Fig. 3.2

3.2.7 Note : It is interesting to note that, if we denote $f(x)$ by y, then from equation (1) we get

$$\left(x+\frac{b}{2a}\right)^2 = \frac{1}{a}\left(y-\frac{4ac-b^2}{4a}\right)$$

which repesents an equation of a parabola whose vertex, focus, axis and directrix are given by

$$\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right), \left(-\frac{b}{2a}, \frac{4ac-b^2+1}{4a}\right), x = -\frac{b}{2a} \text{ and } y = \frac{4ac-b^2-1}{4a}$$

respectively. These concepts wll be discussed geometrically in detail in Mathematics II-B Text Book.

3.2.8 Note : When $a, b, c \in \mathbf{R}$, the minimum or the maximum values of the quadratic expression $f(x) = ax^2 + bx + c$ can be found by using method of calculus.

On differentiating $f(x)$, we get $f'(x) = 2ax + b$ and $f''(x) = 2a\ \forall\ x \in \mathbf{R}$.

Now $f'(x) = 0 \Leftrightarrow 2ax + b = 0 \Leftrightarrow x = -\dfrac{b}{2a}$.

If $a>0$, then $f''(x)>0\ \forall\, x\in\mathbf{R}$ and hence from differentiation rules $f(x)$ has local minimum at $x=-\dfrac{b}{2a}$ and has no local extremum at any other point. Hence $f(x)$ has absolute minimum at $x=-\dfrac{b}{2a}$ and the minumum value of $f(x)$ is $a\left(-\dfrac{b}{2a}\right)^2+b\left(-\dfrac{b}{2a}\right)+c=\dfrac{4ac-b^2}{4a}$.

If $a<0$, then $f''(x)<0\ \forall\, x\in\mathbf{R}$ and hence from differentiation rules $f(x)$ has local maximum at $x=-\dfrac{b}{2a}$ and has no local extremum at any other point. Hence $f(x)$ has absolute maximum at $x=-\dfrac{b}{2a}$ and the maximum value of $f(x)$ is $a\left(-\dfrac{b}{2a}\right)^2+b\left(-\dfrac{b}{2a}\right)+c=\dfrac{4ac-b^2}{4a}$.

3.2.9 Note

(i) When $a,b,c\in\mathbf{R}$, the quadratic expression ax^2+bx+c has no maximum value when $a>0$ and no minimum value when $a<0$.

(ii) When $a,b,c\in\mathbf{R}$ and $a\neq 0$, the curve

$y=ax^2+bx+c$ is symmetric with respect to the line $x=-\dfrac{b}{2a}$.

3.2.10 Example : We find the maximum or minimum (which ever exists) of the expressions

(i) $3x^2+4x+1$ (ii) $4x-x^2-10$

(i) Here $a=3$, $b=4$ and $c=1$.

Since $a=3>0$, the expression $3x^2+4x+1$ has absolute minimum and the minimum value is $\dfrac{4ac-b^2}{4a}=\dfrac{4(3)(1)-(4)^2}{4(3)}=-\dfrac{1}{3}$. This expression has no maximum.

(ii) Here $a=-1$, $b=4$ and $c=-10$.

Since $a=-1<0$, the expression $4x-x^2-10$ has absolute maximum and the maximum value is $\dfrac{4ac-b^2}{4a}=\dfrac{4(-1)(-10)-(4)^2}{4(-1)}=-6$. This expression has no minimum.

3.2.11 Changes in the magnitude (value) of a quadratic expression

Now we discuss the changes in the value of the quadratic expression ax^2+bx+c, when the value of 'x' varies in **R.**

Let $f(x) = ax^2 + bx + c$. This can be written as

$$y = f(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

(i) Suppose that $a > 0$. Then the following are true.

1. If x "approaches $-\infty$", then $f(x)$ "approaches $+\infty$."

2. If $x = -\dfrac{b}{2a}$, then $f(x) = \dfrac{4ac - b^2}{4a}$.

3. If x "approaches $+\infty$" then $f(x)$ "approaches $+\infty$."

4. When x increases from $-\infty$ to $-\dfrac{b}{2a}$, then $f(x)$ decreases from $+\infty$ to $\dfrac{4ac - b^2}{4a}$.

   Also, when x increases from $-\dfrac{b}{2a}$ to $+\infty$, then $f(x)$ increases from $\dfrac{4ac - b^2}{4a}$ to $+\infty$.

(ii) Suppose that $a < 0$. Then the following are true.

1. If x "approaches $-\infty$", then $f(x)$ "approaches $-\infty$."

2. If $x = -\dfrac{b}{2a}$, then $f(x) = \dfrac{4ac - b^2}{4a}$.

3. If x "approaches $+\infty$", then $f(x)$ "approaches $-\infty$."

4. When x increases from $-\infty$ to $-\dfrac{b}{2a}$, then $f(x)$ increases from $-\infty$ to $\dfrac{4ac - b^2}{4a}$.

   Also, when x increases from $-\dfrac{b}{2a}$ to $+\infty$, then $f(x)$ decreases from $\dfrac{4ac - b^2}{4a}$ to $-\infty$.

3.2.12 Solved Problems

1. Problem : *Suppose that the quadratic equations $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root. Then show that $a^3 + b^3 + c^3 = 3abc$.*

Solution : The condition for two quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ to have a common root is

$$(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1).$$

Here $a_1 = a,\ b_1 = b,\ c_1 = c,\ a_2 = b,\ b_2 = c$ and $c_2 = a$.

Therefore $(cb-a^2)^2 = (ac-b^2)(ba-c^2)$

i.e., $b^2c^2 - 2a^2bc + a^4 = a^2bc - ac^3 - b^3a + b^2c^2$

i.e., $a^4 + ab^3 + ac^3 = 3a^2bc$

Hence $a^3 + b^3 + c^3 = 3abc$ (since $a \neq 0$).

2. Problem : *For what values of x the expression $x^2 - 5x - 14$ is positive?*

Solution : Since $x^2 - 5x - 14 = (x+2)(x-7)$, the roots of the equation

$x^2 - 5x - 14 = 0$ are -2 and 7.

Here the coefficient of x^2 is 1, which is positive.

Hence $x^2 - 5x - 14$ is positive when $x < -2$ or $x > 7$ (by Theorem 3.2.3).

3. Problem : *For what values of x the expression $-6x^2 + 2x - 3$ is negative ?*

Solution : $-6x^2 + 2x - 3 = 0$ can be written as $6x^2 - 2x + 3 = 0$.

The roots of this equation are $\dfrac{2 \pm \sqrt{(-2)^2 - 4(6)(3)}}{2(6)}$.

Therefore the roots of $-6x^2 + 2x - 3 = 0$ are non-real complex numbers.

Here the coefficient of x^2 is -6 which is negative.

Hence $-6x^2 + 2x - 3 < 0$ for all $x \in \mathbf{R}$ (by Theorem 3.2.1).

4. Problem : *Find the value of x at which the following expressions have maximum or minimum.*

(i) $x^2 + 5x + 6$ (ii) $2x - x^2 + 7$

Solution

(i) In the expression $x^2 + 5x + 6$, the coefficient of x^2 is positive.

So $x^2 + 5x + 6$ has absolute minimum at $x = -\dfrac{5}{2}$ (since $b = 5, a = 1$).

(ii) In the expression $2x - x^2 + 7$, the coefficient of x^2 is negative.

So $2x - x^2 + 7$ has absolute maximum at $x = -\dfrac{2}{2(-1)} = 1$ (since $b = 2, a = -1$).

5. Problem : *Find the maximum or minimum value of the quadratic expression.*

(i) $2x-7-5x^2$ (ii) $3x^2+2x+11$

Solution

(i) Comparing the given expression with ax^2+bx+c,

we have $a=-5,\ b=2$ and $c=-7$.

So $\dfrac{4ac-b^2}{4a}=\dfrac{4(-5)(-7)-(2)^2}{4(-5)}=\dfrac{-34}{5}$ and $\dfrac{-b}{2a}=\dfrac{-2}{2(-5)}=\dfrac{1}{5}$.

Since $a<0$, $2x-7-5x^2$ has absolute maximum at $x=\dfrac{1}{5}$

and the maximum value is $\dfrac{-34}{3}$.

(ii) Here $a=3,\ b=2$ and $c=11$.

So $\dfrac{4ac-b^2}{4a}=\dfrac{4(3)(11)-(2)^2}{4(3)}=\dfrac{32}{3}$ and $\dfrac{-b}{2a}=\dfrac{-2}{6}=-\dfrac{1}{3}$.

Since $a>0$, $3x^2+2x+11$ has absolute minimum at $x=-\dfrac{1}{3}$

and the minimum value is $\dfrac{32}{3}$.

6. Problem : *Find the changes in the sign of* $4x-5x^2+2$ for $x\in\mathbf{R}$ *and find the extreme value.*

Solution : Comparing the given expression with ax^2+bx+c, we have $a=-5<0$.

The roots of the equation $5x^2-4x-2=0$ are $\dfrac{2\pm\sqrt{14}}{5}$.

Therefore, when $\dfrac{2-\sqrt{14}}{5}<x<\dfrac{2+\sqrt{14}}{5}$, the sign of $4x-5x^2+2$ is positive

and when $x<\dfrac{2-\sqrt{14}}{5}$ or $x>\dfrac{2+\sqrt{14}}{5}$, the sign of $4x-5x^2+2$ is negative.

Since $a<0$, the maximum value of the expression $4x-5x^2+2$ is

$$\frac{4ac-b^2}{4a}=\frac{4(-5)(2)-(4)^2}{4(-5)}=\frac{-56}{-20}=\frac{14}{5}.$$

Hence the extreme value of the expression $4x-5x^2+2$ is $\dfrac{14}{5}$.

7.Problem : *Show that none of the values of the function* $\dfrac{x^2+34x-71}{x^2+2x-7}$ over $\mathbf{R}$ *lies between 5 and 9.*

Solution : Let y_0 be a value of the given function. Then $\exists\, x_0 \in \mathbf{R}$ such that

$$y_0 = \frac{x_0^{\,2}+34x_0-71}{x_0^{\,2}+2x_0-7}.$$

If $y_0 = 1$, then clearly $y_0 \notin (5,9)$.

Suppose that $y_0 \neq 1$. Then the equation $y_0(x^2+2x-7) = x^2+34x-71$ is a quadratic equation and x_0 is a real root of it.

Therefore $(y_0-1)x^2+(2y_0-34)x-(7y_0-71)=0$ is a quadratic equation having a real root x_0. Since all the coefficients of this quadratic equation are real, the other root of the equation is also real.

Therefore $\Delta = (2y_0-34)^2+4(y_0-1)(7y_0-71) \geq 0$.

On simplifying this we get

$$y_0^{\,2}-14y_0+45 \geq 0$$

i.e., $(y_0-5)(y_0-9) \geq 0$

Therefore $y_0 \leq 5$ or $y_0 \geq 9$, Hence y_0 does not lie in $(5,9)$.

Hence none of the values of the given function over $\mathbf{R}$ lies between 5 and 9.

8. Problem : *Find the maximum value of the function* $\dfrac{x^2+14x+9}{x^2+2x+3}$ over $\mathbf{R}$.

Solution : Since the discriminant of x^2+2x+3 is negative, x^2+2x+3 is never zero on $\mathbf{R}$.

Let $x_0 \in \mathbf{R}$ and $y_0 = \dfrac{x_0^{\,2}+14x_0+9}{x_0^{\,2}+2x_0+3}$.

Then $y_0(x_0^{\,2}+2x_0+3) = x_0^{\,2}+14x_0+9.$

i.e., $(1-y_0)x_0^{\,2}+(14-2y_0)x_0+9-3y_0=0.$

Hence x_0 is a root of the equation

$$(1-y_0)x^2+(14-2y_0)x+9-3y_0=0.$$

Since the coefficients of the above equation are real and x_0 is a real root of it, when $y_0 \neq 1$, its discriminant must be ≥ 0.

i.e., $(14-2y_0)^2 - 4(1-y_0)(9-3y_0) \geq 0$

i.e., $-8{y_0}^2 - 8y_0 + 160 \geq 0$.

Hence ${y_0}^2 + y_0 - 20 \leq 0$. i,e., $(y_0+5)(y_0-4) \leq 0$

Therefore $-5 \leq y_0 \leq 4$.

The above inequality is true even when $y_0 = 1$.

Further $y_0 = 4$ when $x_0 = 1$ and $y_0 = -5$ when $x_0 = -2$.

Hence the range of the given function on $\mathbf{R}$ is $[-5,4]$ and the maximum value of the given function is 4.

Exercise 3(b)

I. 1. If the quadratic equations $ax^2 + 2bx + c = 0$ and $ax^2 + 2cx + b = 0, (b \neq c)$ have a common root, then show that $a + 4b + 4c = 0$.

2. If $x^2 - 6x + 5 = 0$ and $x^2 - 12x + p = 0$ have a common root, then find p.

3. If $x^2 - 6x + 5 = 0$ and $x^2 - 3ax + 35 = 0$ have a common root, then find a.

4. If the equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ have a common root and the first equation has equal roots, then prove that $2(b + d) = ac$.

5. Discuss the signs of the following quadratic expressions when x is real.

(i) $x^2 - 5x + 4$ (ii) $x^2 - x + 3$

6. For what values of x, the following expressions are positive?

(i) $x^2 - 5x + 6$ (ii) $3x^2 + 4x + 4$

(iii) $4x - 5x^2 + 2$ (iv) $x^2 - 5x + 14$

7. For what values of x, the following expressions are negative ?

(i) $x^2 - 7x + 10$ (ii) $15 + 4x - 3x^2$

(iii) $2x^2 + 5x - 3$ (iv) $x^2 - 5x - 6$

8. Find the changes in the sign of the following expressions and find their extreme values.

(i) $x^2 - 5x + 6$ (ii) $15 + 4x - 3x^2$

9. Find the maximum or minimum of the following expressions as x varies over $\mathbf{R}$.

(i) $x^2 - x + 7$ (ii) $12x - x^2 - 32$

(iii) $2x + 5 - 3x^2$ (iv) $ax^2 + bx + a$ $(a, b, \in \mathbf{R}$ and $a \neq 0)$

II. 1. Determine the range of the following expressions.

(i) $\dfrac{x^2 + x + 1}{x^2 - x + 1}$ (ii) $\dfrac{x + 2}{2x^2 + 3x + 6}$

(iii) $\dfrac{(x-1)(x+2)}{x+3}$ (iv) $\dfrac{2x^2 - 6x + 5}{x^2 - 3x + 2}$

2. Prove that $\dfrac{1}{3x+1} + \dfrac{1}{x+1} - \dfrac{1}{(3x+1)(x+1)}$ does not lie between 1 and 4, if x is real.

3. If x is real, prove that $\dfrac{x}{x^2 - 5x + 9}$ lies between $-\dfrac{1}{11}$ and 1.

4. If the expression $\dfrac{x - p}{x^2 - 3x + 2}$ takes all real values for $x \in \mathbf{R}$, then find the bounds for p.

5. If $c^2 \neq ab$ and the roots of $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal, then show that $a^3 + b^3 + c^3 = 3abc$ or $a = 0$.

3.3 Quadratic Inequations

Now, we learn the methods of solving quadratic inequations in one variable, having real coefficients.

3.3.1 Definition of quadratic inequation

An inequation of the form $ax^2+bx+c>0$ *or* $ax^2+bx+c\geq 0$ *or* $ax^2+bx+c<0$ *or* $ax^2+bx+c\leq 0$ *where a, b, c are real numbers,* $a\neq 0$ *is called a quadratic inequation in one variable. The set of all values of x which satisfy the given inequation is called the solution set of the inequation.*

3.3.2 Methods of solving inequations

There are two methods for solving inequations.

(i) **Algebraic method :** In this method, we find the solution by factorising the quadratic expression and observing the changes in the sign of the quadratic expression.

(ii) **Graphical method :** In this method, we find the solution from the graph of the inequation.

3.3.3 Example

We solve $x^2-10x+21<0$ by algebraic method and graphical method.

Algebraic Method

We have $x^2-10x+21=(x-3)(x-7)$.

Hence 3 and 7 are the roots of the quadratic equation $x^2-10x+21=0$. Since the coefficient of x^2 is positive, the expression $x^2-10x+21$ is negative iff $3<x<7$.

Therefore the solution set is $\{x\in \mathbf{R} : 3<x<7\}$.

Graphical Method

Let $y=x^2-10x+21$.

The values of y at some selected values of x are given in the following table :

x	0	1	2	3	4	5	6	7	8	9	10
y	21	12	5	0	– 3	– 4	– 3	0	5	12	21

The graph of the function y is drawn using the tabulated values. This is shown in Fig. 3.3. From the figure we observe that

$$y = x^2 - 10x + 21 < 0 \text{ iff } 3 < x < 7.$$

Therefore the solution set is $\{x \in \mathbf{R} : 3 < x < 7\}$.

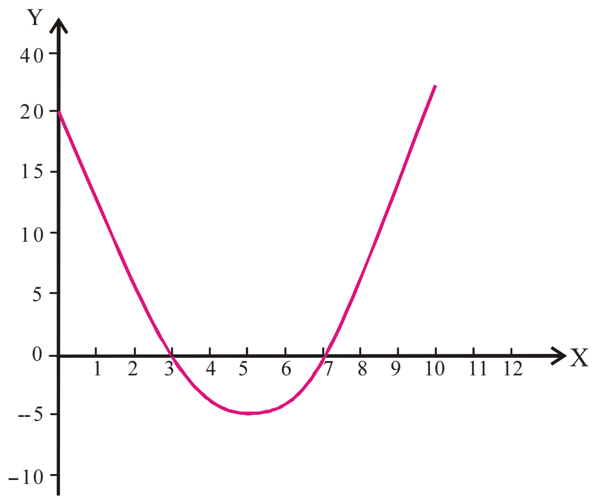


Fig. 3.3

3.3.4 Solved Problems

1. Problem : *Find the solution set of* $x^2 + x - 12 \le 0$ *by both algebraic and graphical methods.*

Solution

Algebraic Method : we have $x^2 + x - 12 = (x+4)(x-3)$.

Hence -4 and 3 are the roots of the equation $x^2 + x - 12 = 0$.

Since the coefficient of x^2 in the quadratic expression $x^2 + x - 12 = 0$ is positive, $x^2 + x - 12$ is negative if $-4 < x < 3$ and positive if either $x < -4$ or $x > 3$.

Hence $x^2 + x - 12 \le 0 \Leftrightarrow -4 \le x \le 3$.

Therefore the solution set is $\{x \in \mathbf{R} : -4 \le x \le 3\}$.

Graphical Method : Let $y = f(x) = x^2 + x - 12$.

The values of y at some selected values of x are given in the following table :

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y = f(x)$	18	8	0	-6	-10	-12	-12	-10	-6	0	8	18

The graph of the function $y = f(x)$ is drawn using the above tabulated values. This is shown in Fig. 3.4.

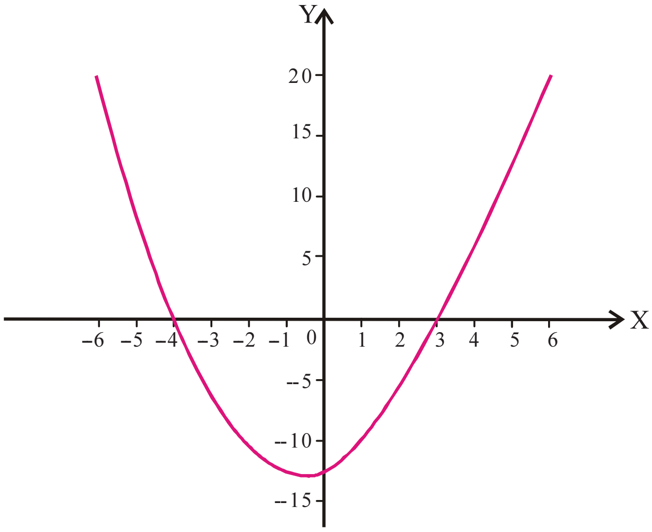


Fig. 3.4

Therefore from the graph of $y = f(x)$ we observe that

$$y = x^2 - x - 12 \le 0 \text{ iff } -4 \le x \le 3.$$

Hence the solution set is $\{x \in \mathbf{R} : -4 \le x \le 3\}$.

2. Problem : *Find the set of values of x for which the inequalities* $x^2 - 3x - 10 < 0,\ 10x - x^2 - 16 > 0$ *hold simultaneously.*

Solution : we have $x^2 - 3x - 10 = (x+2)(x-5)$.

Hence -2 and 5 are the roots of the equation $x^2 - 3x - 10 = 0$.

Since the coefficient of x^2 in the quadratic expression $x^2 - 3x - 10$ is positive

$$x^2 - 3x - 10 < 0 \Leftrightarrow -2 < x < 5.$$

We have $10x - x^2 - 16 = -(x-2)(x-8)$.

Hence 2 and 8 are the roots of the equation $10x - x^2 - 16 = 0$.

Since the coefficient of x^2 in the quadratic expression $10x - x^2 - 16$ is negative

$$10x - x^2 - 16 > 0 \Leftrightarrow 2 < x < 8.$$

Hence $x^2 - 3x - 10 < 0$ and $10x - x^2 - 16 > 0 \Leftrightarrow x \in (-2, 5) \cap (2, 8)$

Therefore the solution set is $\{x \in \mathbf{R} : 2 < x < 5\}$.

3. Problem : *Solve the inequation* $\sqrt{x+2} > \sqrt{8-x^2}$.

Solution : $\sqrt{x+2}, \sqrt{8-x^2}$ are real $\Rightarrow x+2 \geq 0$ and $8 - x^2 \geq 0$

$\Rightarrow x \geq -2$ and $-2\sqrt{2} \leq x \leq 2\sqrt{2}$

$\Rightarrow -2\sqrt{2} \leq x \leq 2\sqrt{2}$ ... (1)

Also

$\sqrt{x+2} > \sqrt{8-x^2}$

$\Rightarrow x+2 > 8-x^2$

$\Rightarrow x^2 + x - 6 > 0$

$\Rightarrow (x+3)\ (x-2) > 0$

$\Rightarrow x < -3$ or $x > 2$ .... (2)

(1) and (2) $\Rightarrow 2 < x \leq 2\sqrt{2}$

Therefore the solution set is $\left\{x \in \mathbf{R} : 2 < x \leq 2\sqrt{2}\right\}$.

4. Problem : *Solve the inequation* $\sqrt{(x-3)(2-x)} < \sqrt{4x^2 + 12x + 11}$.

Solution : The given inequation is equivalent to the following two inequalities.

$(x-3)(2-x) \geq 0$ and $(x-3)(2-x) < 4x^2 + 12x + 11$.

$(x-3)(2-x) \geq 0 \Leftrightarrow (x-2)(x-3) \leq 0$

$\Leftrightarrow 2 \leq x \leq 3$

$-x^2 + 5x - 6 < 4x^2 + 12x + 11 \Leftrightarrow 5x^2 + 7x + 17 > 0$.

The discriminant of the quadratic expression $5x^2 + 7x + 17$ is negative

Hence $5x^2 + 7x + 17 > 0 \ \forall \ x \in \mathbf{R}$.

Hence the solution set of the given inequation is $\{x \in \mathbf{R} : 2 \leq x \leq 3\}$.

5. Problem : *Solve the inequation* $\dfrac{\sqrt{6+x-x^2}}{2x+5} \geq \dfrac{\sqrt{6+x-x^2}}{x+4}$.

Solution : We assume that $x \neq \dfrac{-5}{2}$ and $x \neq -4$

$$\frac{\sqrt{6+x-x^2}}{2x+5} \geq \frac{\sqrt{6+x-x^2}}{x+4}$$

$\Leftrightarrow$ either $6+x-x^2 = 0, 2x+5 \neq 0$ and $x+4 \neq 0$

or $6+x-x^2 > 0$ and $\dfrac{1}{2x+5} \geq \dfrac{1}{x+4}$

We have $\quad 6+x-x^2 = -(x^2-x-6) = -(x+2)(x-3)$

Hence $\quad 6+x-x^2 = 0 \Leftrightarrow x = -2$ or $x = 3$.

$2x+5 = 0 \Leftrightarrow x = -\dfrac{5}{2}$

$x+4 = 0 \Leftrightarrow x = -4$

Therefore $\quad 6+x-x^2 > 0 \Leftrightarrow -2 < x < 3$

For $\quad x \in (-2,3), 2x+5 > -4+5 = 1 > 0$ and $x+4 > -2+4 = 2 > 0$

Hence for $\quad x \in (-2,3), \dfrac{1}{2x+5} \geq \dfrac{1}{x+4} \Leftrightarrow 2x+5 \leq x+4$

$2x+5 \leq x+4 \Leftrightarrow x \leq -1$

Hence $\quad 6+x-x^2 > 0$ and $\dfrac{1}{2x+5} \geq \dfrac{1}{x+4} \Leftrightarrow -2 < x \leq -1$

Therefore the solution set of the given inequation is

$$\{-2,3\} \cup (-2,-1] = [-2,-1] \cup \{3\}.$$

6. Problem : *Solve the inequation* $\sqrt{x^2-3x-10} > (8-x)$.

Solution : $\sqrt{x^2-3x-10} > (8-x) \Leftrightarrow x^2-3x-10 \geq 0$

and (i) $8-x < 0$ or (ii) $x^2-3x-10 > (8-x)^2$ and $8-x \geq 0$

We have $\quad x^2-3x-10 = (x-5)(x+2)$.

Hence $x^2-3x-10\geq 0 \Leftrightarrow x\in(-\infty,-2]\cup[5,\infty)$

$8-x<0 \Leftrightarrow x\in(8,\infty)$

Therefore $x^2-3x-10\geq 0$ and $8-x<0$

$\Leftrightarrow x\in(-\infty,-2]\cup[5,\infty)$ and $x\in(8,\infty)$

$\Leftrightarrow x\in(8,\infty)$.

$$x^2-3x-10>(8-x)^2 \Leftrightarrow x^2-3x-10>64+x^2-16x$$

$$\Leftrightarrow 13x>74$$

$$\Leftrightarrow x>\frac{74}{13} \Leftrightarrow x\in\left(\frac{74}{13},\infty\right)$$

$8-x\geq 0 \Leftrightarrow x\leq 8 \Leftrightarrow x\in(-\infty,8]$.

Hence $x^2-3x-10>(8-x)^2$ and $8-x\geq 0$

$$\Leftrightarrow x\in\left(\frac{74}{13},\infty\right)\cap(-\infty,8]$$

$$\Leftrightarrow x\in\left(\frac{74}{13},8\right].$$

Hence the solution set of the given inequation is

$$(8,\infty)\cup\left(\frac{74}{13},8\right]=\left(\frac{74}{13},\infty\right).$$

Exercise 3(c)

I. 1. Solve the following inequations by algebraic method.

(i) $15x^2+4x-4\leq 0$ (ii) $x^2-2x+1<0$

(iii) $2-3x-2x^2\geq 0$ (iv) $x^2-4x-21\geq 0$

II. 1. Solve the following inequations by graphical method.

(i) $x^2-7x+6>0$ (ii) $4-x^2>0$

(iii) $15x^2+4x-4\leq 0$ (iv) $x^2-4x-21\geq 0$

2. Solve the following inequations.

(i) $\sqrt{3x-8}<-2$ (ii) $\sqrt{-x^2+6x-5}>8-2x$

Key Concepts

- A polynomial of the form $f(x) = ax^2 + bx + c \ (a \neq 0)$ is called a quadratic expression.
- Any equation of the form $ax^2 + bx + c = 0 \quad (a \neq 0)$ is called a quadratic equation in x.
- The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- $\Delta = b^2 - 4ac$ is called the discriminant of the quadratic equation $ax^2 + bx + c = 0$.

  Let α, β be the roots of the quadratic equation $ax^2 + bx + c = 0$, where a, b, c are real numbers.

  Case (i) $\Delta = 0 \Leftrightarrow \alpha = \beta = -\dfrac{b}{2a}$ (a repeated root or double root)

  Case (ii) $\Delta > 0 \Leftrightarrow \alpha$ and β are real and distinct.

  Case (iii) $\Delta < 0 \Leftrightarrow \alpha$ and β are non-real complex numbers conjugate to each other.

  Let a, b and c be rational numbers, α and β be the roots of the equation $ax^2 + bx + c = 0$. Then

  (i) α, β are equal rational numbers if $\Delta = 0$.

  (ii) α, β are distinct rational numbers if Δ is the square of a non zero rational number.

  (iii) α, β are conjugate surds if $\Delta > 0$ and Δ is not the square of a rational number.
- If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

  $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$.
- Let $f(x) = ax^2 + bx + c = 0$ be a quadratic equation and α, β are its roots. Then

  (i) if $c \neq 0$ then $\alpha\beta \neq 0$ and $f\left(\dfrac{1}{x}\right) = 0$ is an equation whose roots are $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$.

  (ii) $f(x-k) = 0$ is an equation whose roots are $\alpha + k$ and $\beta + k$.

  (iii) $f(-x) = 0$ is an equation whose roots are $-\alpha$ and $-\beta$.

  (iv) $f\left(\dfrac{x}{k}\right) = 0$ is an equation whose roots are $k\alpha$ and $k\beta$.
- If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, with $\alpha < \beta$, then

  (i) for $\alpha < x < \beta$, $ax^2 + bx + c$ and a have opposite signs.

  (ii) for $x < \alpha$ or $x > \beta$, $ax^2 + bx + c$ and a have the same sign.
- If $a < 0$, the expression $ax^2 + bx + c$ has maximum at $x = -\dfrac{b}{2a}$ and the maximum value is given by $\dfrac{4ac - b^2}{4a}$.
- If $a > 0$, the expression $ax^2 + bx + c$ has minimum at $x = -\dfrac{b}{2a}$ and the minimum value is given by $\dfrac{4ac - b^2}{4a}$.

Historical Note

Ancient Indian mathematicians did make commendable contribution to geometry, algebra, arithmetic and astronomy. They recognised that quadratic equations have two roots and included negative as well as irrational roots. They could not however solve all quadratics since they did not have the concept of square roots of negative numbers. In indeterminate equations they advanced beyond *Diophantus* (ca.250-334). They had ideas about indeterminate quadratic equations also.

Algebra, as a tool for solving numerical problems, had been brought to an advanced stage of development as early as 2000 B.C. by the *Babylonians*. The development of a system of algebraic symbols, however, was not begun until the 15th century A.D. The development of this system - which amounts to a mathematical "shorthand" - has made possible huge advances in mathematics.

Prior to 17th century, the theory of equations was handicapped by the failure of mathematicians to recognise negative or complex roots. Only Indian mathematicians, *Brahmagupta* (ca.589-668) recognised negative roots. Arab mathematicians who made notable contributions to the theory of equations included *al-Khowarizmi* (8th century) and *Omar Khayyam* (ca.1044-1123) who developed first the theory of cubic equations.

In 1545, the Italian mathematician *Girolamo Cardano* (1501-1576) published an algebraic solution to cubic equation. *Nicolo Tartaglia* (1500-1557) too contributed greatly to the subject.

L. Ferrari (1522-1565) solved fourth degree equation algebraically. In 1635 *Rene Descartes* (1596-1650) published the rule of signs for the number of positive and negative roots of an equation. A few decades later *Isaac Newton* (1642-1727) gave an iterative method.

At the end of the 18th century *C.F.Gauss* (1777-1855) proved that every polynomial equation has atleast one root.

Evariste Galois (1811-1832) established the fact that a polynomial could be solved by means of algebraic formula for equations of degree less than five only.

Other notable mathematicians who contributed for the development of algebra are : *Aryabhatta* (6th century), *Bhaskara-I* (7th century), *Mahaveeracharya* (9th century), *Euler* (1707-1783), *J.L. Lagrange* (1736-1813), *George Boole* (1815-1864), and *John Von Neumann* (1906-1993) and several others.

Answers

Exercise 3(a)

I. 1. (i) 4, 3 (ii) $-1, 2$ (iii) $\dfrac{-3 \pm i\sqrt{7}}{4}$ (iv) $\dfrac{2}{\sqrt{3}}, -4\sqrt{3}$ (v) $\dfrac{\sqrt{5}}{2}, -\dfrac{1}{\sqrt{5}}$

2. (i) $x^2 - 7x + 10 = 0$ (ii) $mnx^2 + (n^2 - m^2)x - mn = 0$

(iii) $(p^2 - q^2)x^2 + 4px - p^2 + q^2 = 0$ (iv) $x^2 - 14x + 29 = 0$ (v) $x^2 + 6x + 34 = 0$

3. (i) real and distinct (ii) rational and equal

(iii) rational and distinct (iv) conjugate complex numbers

4. (i) $-\dfrac{b}{c}$ (ii) $\dfrac{b^2 - 2ac}{c^2}$ (iii) $\dfrac{bc^4(3ac - b^2)}{a^7}$ (iv) $\dfrac{b^2(b^2 - 4ac)}{a^2c^2}$ (v) $\dfrac{c^2}{a^2}$

5. (i) 3, 5 (ii) $\dfrac{1}{3}$ (iii) $-5, 3$ (iv) $-\dfrac{1}{2}, 1$ (v) $\dfrac{-5 \pm \sqrt{41}}{2}$

6. $x^2 - 2(p^2 - 2q)x + p^2(p^2 - 4q) = 0$ 9. $nb^2 = (1+n)^2 ac$ 10. 12, 14

II. 1. $\dfrac{b^2 - 2ac}{a^2c^2}$ 2. $a^2c^2x^2 - (b^2 - 2ac)(a^2 + c^2)x + (b^2 - 2ac)^2 = 0$

3. $\left\{\dfrac{-3 \pm \sqrt{5}}{2}, \dfrac{1}{2}, 2\right\}$ 4. $\{-1, 1\}$ 5. $\{1, 2\}$ 6. $\{-1, 4\}$

7. $\left\{\dfrac{1}{2}\right\}$ 8. $\left\{\dfrac{1}{2}, \dfrac{1 \pm i\sqrt{3}}{2}, 2\right\}$ 9. $\left\{\dfrac{1 \pm i\sqrt{3}}{2}, 2 \pm \sqrt{3}\right\}$

10. $x^2 - 7x + 12 = 0$

Exercise 3(b)

I. 2. 11 or 35 3. $a = 4$ or 12

5. (i) Positive when $x \in (-\infty, 1) \cup (4, \infty)$ and negative when $x \in (1, 4)$.

(ii) Positive for $x \in \mathbf{R}$

6. (i) $x < 2$ or $x > 3$ (ii) $\forall x \in \mathbf{R}$

(iii) $\dfrac{2 - \sqrt{14}}{5} < x < \dfrac{2 + \sqrt{14}}{5}$ (iv) $\forall x \in \mathbf{R}$

7. (i) $2 < x < 5$ (ii) $x < -\frac{5}{3}$ or $x > 3$

(iii) $-3 < x < \frac{1}{2}$ (iv) $-1 < x < 6$

8. (i) If $2 < x < 3$, the expression is negative

If $x < 2$ or $x > 3$, the expression is positive. The extreme values is $-\frac{1}{4}$

(ii) If $\frac{-5}{3} < x < 3$, the expression is positive

If $x < \frac{-5}{3}$ or $x > 3$, the expression is negative. The extreme values is $\frac{49}{3}$

9. (i) $\frac{27}{4}$ is the minimum value (ii) 4 is the maximum value

(iii) $\frac{16}{3}$ is the maximum value

(iv) $\frac{4a^2 - b^2}{4a}$ is the maximum value for $a < 0$ and minimum value for $a > 0$.

II. 1. (i) $\left[\frac{1}{3}, 3\right]$ (ii) $\left[-\frac{1}{13}, \frac{1}{3}\right]$

(iii) $(-\infty, -9] \cup [-1, \infty)$ (iv) $(-\infty, -2] \cup [2, \infty)$

4. $1 < p < 2$

Exercise 3(c)

I. 1. (i) $-\frac{2}{3} \le x \le \frac{2}{5}$ (ii) Solution does not exist

(iii) $-2 \le x \le \frac{1}{2}$ (iv) $x \le -3$ or $x \ge 7$.

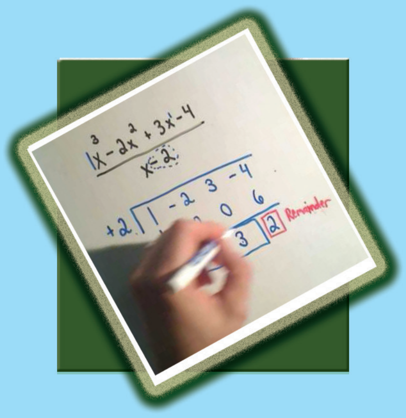
II. 1. (i) The values of x left to 1 and right to +6.

(ii) The values of x which lie between -2 and +2.

(iii) The values of x lie between $-\frac{2}{3}$ and $+\frac{2}{5}$, including $-\frac{2}{3}$ and $+\frac{2}{5}$.

(iv) The values of x lie left side to -3 and right side to +7, including -3 and +7.

2. (i) Solution does not exist (ii) $\{x : 3 < x \le 5\}$



Chapter 4

Theory of Equations

" Mathematics is the science of what is clear by itself " - ***Carl Jacobi***

Introduction

In the earlier classes, we have studied the linear equations in one and two variables, the rational integral function of x as a polynomial in one variable and the solutions of quadratic equations and inequations.

In the previous chapter, we have learnt about the quadratic expressions, quadratic equations and inequations more in detail. We have established certain relations between the roots and coefficients of quadratic equations. But in many problems that arise in science and technology we encounter equations of degree higher than two. We now investigate the relations which hold between the roots and the coefficients of equations of the n^{th} degree and then discuss some elementary properties in the general theory of equations.

Pavuluri Mallana

(11^{th} Century)

Pavuluri Mallana *of 11^{th} century A.D. was a mathematician - poet of repute, who wrote his magnum opus in Telugu (Prosodical form) and named it Sarasangraha Ganitham which is in more than one way not mere a translation, but transcreation of Mahaveeracharya's (9th century A.D.) Ganitha Sangraha Sara. Mallana, it seems wrote 10 chapters, but only three chapters are available. These contain mostly arithmetic and some elementary algebra dealing with linear and quadratic equations.*

4.1 Relation between the roots and the coefficients in an equation

In this section we define a polynomial of degree n and derive the relation between roots and coefficients of the polynomial equation.

4.1.1 Definitions

If n is a non-negative integer and $a_0, a_1, a_2, ..., a_n$ are real or complex numbers and $a_0 \neq 0$, then an expression

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n \qquad ... (1)$$

is called a ***polynomial in x of degree n***.

Here $a_0, a_1, ..., a_n$ are called the coefficients of the polynomial $f(x)$, while a_0 is called the leading coefficient of $f(x)$, a_n is called the constant term or absolute term of $f(x)$ and a_i is called the coefficient of x^{n-i}.

Some times a polynomial in x of degree n is represented by

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. If $a_n \neq 0$ then a_n is the leading coefficient of $f(x)$, a_0 is the constant (absolute) term of $f(x)$ and a_i is the coefficient of x^i.

By definition $a_0 x^n + a_1 x^{n-1} + \cdots + a_n = b_0 x^n + b_1 x^{n-1} + \cdots + b_n$ if and only if

$$a_k = b_k, \ \forall \ k = 0, 1, 2, ..., n.$$

We note that a non-zero constant is a polynomial of degree zero. The constant zero is called the zero polynomial and its degree is not defined.

A polynomial with leading coefficient 1 is called a ***monic polynomial***.

Any polynomial $p(x)$ of degree $n \geq 1$ can be written as $p(x) = a_0 p_0(x)$, where a_0 is the leading coefficient of $p(x)$ and $p_0(x)$ is a monic polynomial of degree n.

If $f(x)$ is a polynomial of degree $n > 0$, then the equation $f(x) = 0$ is called a polynomial equation of degree n. It is also called as an algebraic equation of degree n.

If $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ is a polynomial and α is a real or complex number, then we write $f(\alpha)$ for $a_0 \alpha^n + a_1 \alpha^{n-1} + \cdots + a_n$.

A real or complex number α is said to be a ***zero*** *of a polynomial $f(x)$ or a* ***root*** *of the equation $f(x) = 0$, if $f(\alpha) = 0$.*

If $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ *and* $g(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m$ *are polynomials of degree n and m respectively with* $n \le m$*, without loss of generality then we can define the addition and multiplication of polynomials which satisfy all the arithmetical properties of real numbers (or complex numbers), except the existence of reciprocals of nonzero polynomial.*

The main object of the theory of equations is to find roots of a polynomial equation $f(x) = 0$, i.e., to solve the equation $f(x) = 0$.

A real number can be reckoned as a complex number whose imaginary part is zero. So $\mathbf{R} \subseteq \mathbf{C}$. Here onwards we consider only the set of complex numbers **C** in our discussions.

4.1.2 Theorem (Remainder Theorem) : *Let* $f(x)$ *be a polynomial of degree* $n > 0$. *Let* $a \in \mathbf{C}$. *Then there exists a polynomial* $q(x)$ *of degree* $n-1$ *such that*

$$f(x) = (x-a)\, q(x) + f(a).$$

Proof : Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$.

Now define

$$b_0 = a_0, \quad b_1 = a_1 + ab_0$$

$$b_2 = a_2 + ab_1, \ldots, b_{n-1} = a_{n-1} + ab_{n-2}$$

and $\quad q(x) = b_0 x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-2} x + b_{n-1}$.

Then, $(x-a)\, q(x) = x\, q(x) - a\, q(x)$

$$= x(b_0 x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-2} x + b_{n-1})$$

$$- a(b_0 x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-2} x + b_{n-1})$$

$$= b_0 x^n + (b_1 - ab_0) x^{n-1} + \cdots + (b_{n-1} - ab_{n-2}) x - ab_{n-1}$$

$$= a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x - ab_{n-1}$$

$$= f(x) - a_n - ab_{n-1}$$

By substituting $x = a$

we get $\quad (a-a)\, q(a) = f(a) - a_n - ab_{n-1}$

$\Rightarrow \; 0 = f(a) - a_n - ab_{n-1}$

$\Rightarrow \; f(a) = a_n + ab_{n-1}$

$\therefore \; (x-a)\, q(x) = f(x) - f(a)$

i.e., $\; f(x) = (x-a)\, q(x) + f(a)$.

Therefore $f(x) = (x-a)\,q(x) + f(a)$.

4.1.3 Definition

Let $f(x)$ be a polynomial of degree $n > 0$. Let $a \in \mathbf{C}$. We say that $x-a$ is a factor of $f(x)$, if there exists a polynomial $q(x)$ such that $f(x) = (x-a)\,q(x)$.

4.1.4 Corollary : Let $f(x)$ be a polynomial of degree $n > 0$. Then $x-a$ is a factor of $f(x)$ iff $f(a) = 0$.

4.1.5 Theorem (Fundamental Theorem of Algebra) : *Every non constant polynomial equation has atleast one root.*

(The proof of this theorem is beyond the scope of this book.)

4.1.6 Theorem : *The set of all roots of a polynomial equation $f(x) = 0$ of degree $n > 0$ is non empty and has atmost n elements. Also there exist $\alpha_1, \alpha_2, ..., \alpha_n$ (which may not be distinct) in $\mathbf{C}$ such that $f(x) = a(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$, where a is the leading coefficient of $f(x)$.*

Proof : We prove the theorem by induction. The result is true when $n = 1$ (since in this case, $f(x) = a(x-\alpha)$ for some α).

Assume the truth of the theorem for $n-1$ where $n \geq 2$. Now suppose that $f(x)$ is a polynomial of degree n with leading coefficient a. Then by the fundamental theorem of algebra $f(x)$ has atleast one root, say α_1. Then, by Corollary 4.1.4, $x-\alpha_1$ is a factor of $f(x)$ and

$$f(x) = (x-\alpha_1)\,q(x) \qquad \text{... (1)}$$

where $q(x)$ is a polynomial of degree $n-1$. Since the leading coefficient of $f(x)$ is a, the leading coefficient of $q(x)$ is also a. Now, by the induction hypothesis, there exist $\alpha_2, ..., \alpha_n$ in $\mathbf{C}$ such that

$$q(x) = a(x-\alpha_2)...(x-\alpha_n) \qquad \text{... (2)}$$

From (1) and (2), we have

$$f(x) = (x-\alpha_1)\,q(x) = a(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n).$$

4.1.7 Note

(i) From Theorem 4.1.6 it follows that a polynomial equation of degree $n > 0$ has atmost n distinct roots.

(ii) Let $f(x)$ be a polynomial of degree $n > 0$ with leading coefficient a.

If $\alpha_1, \alpha_2, ..., \alpha_n$ and $\beta_1, \beta_2, ..., \beta_n$ are complex numbers such that

$$f(x) = a(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$$
$$= a(x-\beta_1)(x-\beta_2)...(x-\beta_n),$$

then using the principle of mathematical induction, it can be shown that

$(\beta_1, \beta_2, ..., \beta_n)$ is a permutation of $(\alpha_1, \alpha_2, ..., \alpha_n)$.

(iii) If $f(x)$ is a polynomial of degree $n > 0$ with leading coefficient a and $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbf{C}$ are such that

$$f(x) = a\ (x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$$

then $\alpha_1, \alpha_2, ..., \alpha_n$ are the roots of the polynomial equation $f(x) = 0$.

For example if $f(x) = (x-1)^2 (x-2)$, then 1, 1, 2 are the roots of $f(x) = 0$.

4.1.8 Corollary : Suppose n is a positive integer, $a_0, a_1, ..., a_n$ and $b_0, b_1, ..., b_n$ are complex numbers such that

$$a_0\alpha^n + a_1\alpha^{n-1} + \cdots + a_n = b_0\alpha^n + b_1\alpha^{n-1} + \cdots + b_n$$

for more than n distinct elements α in $\mathbf{R}$. Then

$$a_k = b_k \quad \text{for} \quad 0 \le k \le n$$

Proof : Let $h(x) = (a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_n - b_n)$ be a nonzero polynomial.

By hypothesis, $h(x) = 0$ has more than n roots.

By Theorem 4.1.6, $h(x)$ is a non zero constant polynomial and therefore number of roots of $h(x) = 0$ is zero. Since this contradicts the hypothesis that $h(x) = 0$ has more than n roots, it follows that $h(x)$ is the zero polynomial. Accordingly, $a_k = b_k$ for $k = 0, 1, 2, ..., n$.

4.1.9 Note : If $f(x)$ and $g(x)$ are polynomials such that $f(\alpha) = g(\alpha)$ for infinitely many numbers α, then $f(x) = g(x)$.

4.1.10 The relations between the roots and the coefficients

Let us consider the n^{th} degree polynomial equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0.$$

Let $\alpha_1, \alpha_2, ..., \alpha_n$ be its roots.

Then we have

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)$$
$$= x^n - (\alpha_1+\alpha_2+\cdots+\alpha_n)x^{n-1}$$
$$+ (\alpha_1\alpha_2+\alpha_2\alpha_3+\cdots+\alpha_{n-1}\alpha_n)x^{n-2}$$
$$-\cdots+(-1)^n\,\alpha_1\alpha_2\cdots\alpha_n.$$

On equating the coefficients of like powers of x in this equation and denoting the sum of products of the roots taken r at a time by s_r, we get

$$-p_1 = s_1 = \sum_{i=1}^{n} \alpha_i \text{ (sum of the roots)},$$

$$p_2 = s_2 = \sum_{1\le i<j\le n} \alpha_i\alpha_j \text{ (sum of the products of the roots taken two at a time)}$$

$$-p_3 = s_3 = \sum_{1\le i<j<k\le n} \alpha_i\alpha_j\alpha_k \text{ (sum of the products of the roots taken three at a time)}$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$(-1)^n p_n = s_n = \alpha_1\alpha_2\alpha_3\ldots\alpha_n \text{ (product of the roots).}$$

These equalities give the relations between the roots and the coefficients for any polynomial equation whose leading coefficient is 1.

4.1.11 Note

(i) If the leading coefficient in $f(x)$ is a_0 then on dividing each term of the equation $f(x) = 0$ by $a_0 \neq 0$, we get

$$x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \cdots + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0} = 0$$

whose roots coincide with those of $f(x) = 0$.

In this case, the above relations reduce to

$$s_1 = -\frac{a_1}{a_0},\ s_2 = \frac{a_2}{a_0}, \cdots,\ s_n = (-1)^n\frac{a_n}{a_0};\ \text{i.e., } s_r = (-1)^r\frac{a_r}{a_0} \text{ for } 1\le r\le n.$$

(ii) We recall that in the case of quadratic equation

$$ax^2+bx+c=0 \text{ with roots } \alpha \text{ and } \beta,\quad \alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

(iii) For $n = 3$, we get a cubic equation $x^3 + p_1x^2 + p_2x + p_3 = 0$.

Let α_1, α_2 and α_3 be its roots.

Then $s_1 = \alpha_1+\alpha_2+\alpha_3 = -p_1,$

$$s_2 = \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 = p_2$$

and $\quad s_3 = \alpha_1 \alpha_2 \alpha_3 = -p_3$.

(iv) For $n = 4$, we get a biquadratic equation $x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$.

Let $\alpha_1, \alpha_2, \alpha_3$ and α_4 be its roots.

Then $\quad s_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -p_1,$

$s_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_4 = p_2,$

$s_3 = \alpha_1 \alpha_2 \alpha_3 + \alpha_2 \alpha_3\alpha_4 + \alpha_3\alpha_4\alpha_1 + \alpha_1 \alpha_2 \alpha_4 = -p_3$

and $\quad s_4 = \alpha_1 \alpha_2 \alpha_3 \alpha_4 = p_4$.

(v) If $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbf{C}$ and $p_1, p_2, \cdots, p_n$ are defined by

$$p_r = (-1)^r \sum_{1 \le i_1 < i_2 \ldots < i_r \le n} \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_r} \; (r = 1, 2, \ldots, n),$$

then $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the roots of $x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0$.

4.1.12 Solved Problems

1. Problem : *Form the monic polynomial equation of degree 3 whose roots are 2,3* and 6.

Solution : The required monic polynomial equation is $(x-2)(x-3)(x-6) = 0$.

On simplification, this reduces to $x^3 - 11x^2 + 36x - 36 = 0$.

2. Problem : *Find the relations between the roots and the coefficients of the cubic equation*

$3x^3 - 10x^2 + 7x + 10 = 0$.

Solution : Given cubic equation is $3x^3 - 10x^2 + 7x + 10 = 0$.

On dividing the equation by 3, we get

$$x^3 - \frac{10}{3}x^2 + \frac{7}{3}x + \frac{10}{3} = 0. \qquad \ldots (1)$$

On comparing (1) with $x^3 + p_1x^2 + p_2x + p_3 = 0$, we have

$$p_1 = -\frac{10}{3},\; p_2 = \frac{7}{3} \text{ and } p_3 = \frac{10}{3}.$$

Let α, β, γ be the roots of (1). Then

$$\Sigma\alpha = -p_1 = -\left(-\frac{10}{3}\right) = \frac{10}{3},$$

$$\Sigma\alpha\beta = p_2 = \frac{7}{3},$$

and $\alpha\beta\gamma = -p_3 = -\frac{10}{3}.$

3. Problem : *Write down the relations between the roots and the coefficients of the biquadratic equation* $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0.$

Solution : Given equation is $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0.$ ... (1)

On comparing (1) with $x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$, we have

$$p_1 = -2,\ p_2 = 4,\ p_3 = 6 \text{ and } p_4 = -21.$$

Let $\alpha, \beta, \gamma, \delta$ be the roots of (1). Then,

$$\Sigma\alpha = -p_1 = 2,$$

$$\Sigma\alpha\beta = p_2 = 4,$$

$$\Sigma\alpha\beta\gamma = -p_3 = -6$$

and $\alpha\beta\gamma\delta = p_4 = -21.$

4. Problem : *If 1,2,3 and 4 are the roots of the equation* $x^4 + ax^3 + bx^2 + cx + d = 0$, *then find the values of a, b, c and d.*

Solution : Given that the roots of the polynomial equation are 1,2,3 and 4, then

$$x^4 + ax^3 + bx^2 + cx + d = (x-1)(x-2)(x-3)(x-4) = 0.$$

i.e., $x^4 + ax^3 + bx^2 + cx + d = x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$

On equating the coefficients of like powers of x, we obtain

$$a = -10,\ b = 35,\ c = -50 \text{ and } d = 24.$$

5. Problem : *If a, b, c are the roots of the equation* $x^3 - px^2 + qx - r = 0$ *and* $r \neq 0$, *then find* $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$ *in terms of* p, q, r.

Solution : Given that a, b, c are the roots of $x^3 - px^2 + qx - r = 0$ and $r \neq 0$. Then, by Note 4.1.11 (iii)

$$a + b + c = p,\ ab + bc + ca = q \text{ and } abc = r.$$

Since $r \neq 0$, it follows that none of a, b, c is zero.

$$\text{Now, } \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{b^2c^2 + a^2c^2 + a^2b^2}{a^2b^2c^2}$$

$$= \frac{(ab + bc + ca)^2 - 2abc(a + b + c)}{a^2b^2c^2} = \frac{q^2 - 2rp}{r^2}.$$

6. Problem : *Find the sum of the squares and the sum of the cubes of the roots of the equation* $x^3 - px^2 + qx - r = 0$ *in terms of* p, q, r.

Solution : Let a, b, c be the roots of the given equation.

Then $a+b+c = p$, $ab+bc+ca = q$ and $abc = r$.

Sum of the squares of the roots is

$$a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca) = p^2 - 2q.$$

Sum of the cubes of the roots is

$$a^3 + b^3 + c^3 = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) + 3abc$$

$$= p(p^2 - 2q - q) + 3r = p(p^2 - 3q) + 3r.$$

7. Problem : *Obtain the monic cubic equation, whose roots are the squares of the roots of the equation* $x^3 + p_1x^2 + p_2x + p_3 = 0$.

Solution : Let a, b, c be the roots of the given equation

$$x^3 + p_1x^2 + p_2x + p_3 = 0.$$

Then $a+b+c = -p_1, ab+bc+ca = p_2$ and $abc = -p_3$. ... (1)

Let $-s_1 = a^2+b^2+c^2$, $s_2 = a^2b^2 + b^2c^2 + c^2a^2$ and $-s_3 = a^2b^2c^2$. ... (2)

Then $x^3 + s_1x^2 + s_2x + s_3 = (x-a^2)(x-b^2)(x-c^2)$.

Hence a^2, b^2, c^2 are the roots of the equation $x^3 + s_1x^2 + s_2x + s_3 = 0$.

Now from (1) and (2), we get

$$-s_1 = (a+b+c)^2 - 2(ab+bc+ca) = (-p_1)^2 - 2p_2 = p_1^2 - 2p_2$$

$$s_2 = (ab+bc+ca)^2 - 2(a^2bc + b^2ca + c^2ab)$$

$$= p_2^2 - 2abc(a+b+c)$$

$$= p_2^2 - 2.(-p_3).(-p_1) = p_2^2 - 2p_1p_3$$

and $-s_3 = (abc)^2 = (-p_3)^2 = p_3^2$.

Hence $s_1 = 2p_2 - p_1^2$, $s_2 = p_2^2 - 2p_1p_3$ and $s_3 = -p_3^2$.

Therefore the required equation is

$$x^3 + (2p_2 - p_1^2)\,x^2 + (p_2^2 - 2p_1p_3)\,x - p_3^2 = 0.$$

4.1.13 Notation : If α, β and γ are the roots of a polynomial equation of degree 3, then $\alpha+\beta+\gamma$ is denoted by $\Sigma\alpha$, $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ is denoted by $\Sigma\frac{1}{\alpha}$, $\alpha\beta+\beta\gamma+\gamma\alpha$ is denoted by $\Sigma\alpha\beta$ and $\alpha^2\beta+\beta^2\alpha+\alpha^2\gamma+\gamma^2\alpha+\beta^2\gamma+\gamma^2\beta$ is denoted by $\Sigma\alpha^2\beta+\Sigma\alpha\beta^2$. This notation can also be extended to the roots of an n^{th} degree polynomial equation.

4.1.14 Solved Problems

1. Problem : *Let α, β, γ be the roots of the equation $x^3+px^2+qx+r=0$. Then find*

(i) $\Sigma\alpha^2$ (ii) $\Sigma\frac{1}{\alpha}$, *if α, β, γ are non zero*

(iii) $\Sigma\alpha^3$ (iv) $\Sigma\beta^2\gamma^2$ (v) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

Solution : Since α, β, γ are the roots of the given equation, we have

$$\alpha+\beta+\gamma=-p \qquad \text{... (1)}$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=q \qquad \text{... (2)}$$

$$\alpha\beta\gamma=-r \qquad \text{... (3)}$$

(i) To find $\Sigma\alpha^2$.

On squaring equation (1), we get

$$\alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\beta\gamma+2\gamma\alpha=p^2.$$

i.e., $\Sigma\alpha^2+2\Sigma\alpha\beta=p^2$.

i.e., $\Sigma\alpha^2+2q=p^2$.

Therefore $\Sigma\alpha^2=p^2-2q$.

(ii) To find $\Sigma\frac{1}{\alpha}$.

$$\Sigma\frac{1}{\alpha}=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}=-\frac{q}{r}.$$

(iii) To find $\Sigma\alpha^3$.

We know that

$$\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha)$$

$$=(\alpha+\beta+\gamma)\left[(\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha)\right]$$

i.e., $\Sigma\alpha^3 - 3(-r) = -p[p^2 - 3q]$.

$\therefore\ \Sigma\alpha^3 = 3pq - p^3 - 3r$.

(iv) To find $\Sigma\beta^2\gamma^2$

On squaring equation (2), we get

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + 2\alpha\beta\gamma(\alpha+\beta+\gamma) = q^2$$

i.e., $\Sigma\beta^2\gamma^2 + 2pr = q^2$

Therefore $\Sigma\beta^2\gamma^2 = q^2 - 2pr$.

(v) To find $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$.

From equation (1), $\alpha+\beta+\gamma = -p$.

Then $\alpha+\beta = -p-\gamma = -(p+\gamma)$.

Similarly, $\beta+\gamma = -(p+\alpha)$ and $\gamma+\alpha = -(p+\beta)$.

Therefore $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) = -(p+\gamma)(p+\alpha)(p+\beta)$

$$= -p^3 - p^2(\alpha+\beta+\gamma) - p(\alpha\beta+\beta\gamma+\gamma\alpha) - \alpha\beta\gamma$$

$$= -p^3 + p^3 - pq + r$$

$$= r - pq.$$

2. Problem : *If α, β, γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$, then find* $\Sigma\alpha^2\beta + \Sigma\alpha\beta^2$.

Solution : Since α, β and γ are the roots of the given equation, we have

$$\alpha+\beta+\gamma = -a \qquad \dots (1)$$

$$\alpha\beta+\beta\gamma+\gamma\alpha = b \qquad \dots (2)$$

$$\alpha\beta\gamma = -c \qquad \dots (3)$$

On multiplying (1) and (2), we get

$$\Sigma\alpha^2\beta + \Sigma\alpha\beta^2 + 3\alpha\beta\gamma = -ab$$

i.e., $\Sigma\alpha^2\beta + \Sigma\alpha\beta^2 - 3c = -ab$ (by equation (3))

Therefore $\Sigma\alpha^2\beta + \Sigma\alpha\beta^2 = 3c - ab$.

3. Problem : *If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, then form the monic cubic equation, whose roots are $\alpha(\beta+\gamma), \beta(\gamma+\alpha), \gamma(\alpha+\beta)$.*

Solution : Since α, β and γ be the roots of the given equation, we have

$$\alpha+\beta+\gamma=-p, \quad \text{... (1)}$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=q \quad \text{... (2)}$$

$$\alpha\beta\gamma=-r \quad \text{... (3)}$$

Let $\Sigma\alpha(\beta+\gamma) = s_1$

$\Sigma\alpha\beta\,(\beta+\gamma)(\gamma+\alpha) = s_2$

and $\alpha\beta\gamma(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta) = s_3$

Then the required equation is $x^3 - s_1x^2 + s_2x - s_3 = 0$.

Now we find s_1, s_2 and s_3 in terms of p, q and r.

$$s_1 = \Sigma\alpha(\beta+\gamma) = \Sigma(\alpha\beta+\alpha\gamma)$$

$$= \Sigma\alpha\beta+\Sigma\alpha\gamma = \Sigma\alpha\beta+\Sigma\alpha\beta = 2\Sigma\alpha\beta = 2q$$

$$s_2 = \Sigma\alpha\beta(\beta+\gamma)(\gamma+\alpha) = \Sigma\alpha\beta(-p-\alpha)(-p-\beta) \text{ (by (1))}$$

$$= \Sigma\alpha\beta\left\{p^2+(\alpha+\beta)\,p+\alpha\beta\right\}$$

$$= \Sigma\alpha\beta\left\{p^2-(p+\gamma)\,p+\alpha\beta\right\}$$

$$= \Sigma\alpha\beta\{-\gamma p+\alpha\beta\} = \Sigma(\alpha^2\beta^2-\alpha\beta\gamma p)$$

$$= \Sigma\alpha^2\beta^2 - p\Sigma\alpha\beta\gamma = \Sigma\alpha^2\beta^2 - p(-3r)$$

$$= (\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)+3pr$$

$$= q^2-2pr+3pr = q^2+pr$$

$$s_3 = \alpha\beta\gamma(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$$

$$= -r(-p-\gamma)(-p-\alpha)(-p-\beta)$$

$$= r(p+\gamma)(p+\alpha)(p+\beta)$$

$$= r[p^3+(\alpha+\beta+\gamma)\,p^2+(\alpha\beta+\beta\gamma+\gamma\alpha)\,p+\alpha\beta\gamma]$$

$$= r[p^3-p^3+pq-r] = r(pq-r).$$

Therefore the required equation is

$$x^3-2qx^2+(q^2+pr)\,x-r(pq-r)=0.$$

Exercise 4(a)

I. 1. Form polynomial equations of the lowest degree, with roots as given below :

(i) $1, -1, 3$ (ii) $1 \pm 2i, 4, 2$

(iii) $2 \pm \sqrt{3}, 1 \pm 2i$ (iv) $0, 0, 2, 2, -2, -2$

(v) $1 \pm \sqrt{3}, 2, 5$ (vi) $0, 1, \frac{-3}{2}, \frac{-5}{2}$

2. If α, β, γ are the roots of the equation $4x^3 - 6x^2 + 7x + 3 = 0$, then find the value of $\alpha\beta + \beta\gamma + \gamma\alpha$.

3. If $1, 1, \alpha$ are the roots of the equation $x^3 - 6x^2 + 9x - 4 = 0$, then find α.

4. If $-1, 2$ and α are the roots of the equation $2x^3 + x^2 - 7x - 6 = 0$, then find α.

5. If $1, -2$ and 3 are the roots of the equation $x^3 - 2x^2 + ax + 6 = 0$, then find a.

6. If the product of the roots of the equation $4x^3 + 16x^2 - 9x - a = 0$ is 9, then find a.

7. Find s_1, s_2, s_3 and s_4 for each of the following equations.

(i) $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$ (ii) $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$

$$\left[\text{Hint : } s_1 = \sum_{i=1}^{4} \alpha_i,\ s_2 = \sum_{1 \le i < j \le 4} \alpha_i \alpha_j,\ s_3 = \sum_{1 \le i < j < k \le 4} \alpha_i \alpha_j \alpha_k,\ s_4 = \alpha_1\alpha_2\alpha_3\alpha_4 \right]$$

II. 1. If α, β and 1 are the roots of the equation $x^3 - 2x^2 - 5x + 6 = 0$, then find α and β.

2. If α, β and γ are the roots of the equation $x^3 - 2x^2 + 3x - 4 = 0$, then find

(i) $\Sigma\alpha^2\beta^2$ (ii) $\Sigma\alpha\beta(\alpha + \beta)$

3. If α, β and γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, then find

(i) $\Sigma \frac{1}{\alpha^2\beta^2}$ (ii) $\frac{\beta^2 + \gamma^2}{\beta\gamma} + \frac{\gamma^2 + \alpha^2}{\gamma\alpha} + \frac{\alpha^2 + \beta^2}{\alpha\beta}$

(iii) $(\beta + \gamma - 3\alpha)(\gamma + \alpha - 3\beta)(\alpha + \beta - 3\gamma)$ (iv) $\Sigma\alpha^3\beta^3$

III. 1. If α, β, γ are the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$, then find the equation whose roots are $\alpha^2 + \beta^2, \beta^2 + \gamma^2, \gamma^2 + \alpha^2$.

2. If α, β, γ are the roots of the equation $x^3 - 7x + 6 = 0$, then find the equation whose roots are $(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$.

3. If α, β, γ are the roots of the equation $x^3 - 3ax + b = 0$, then prove that $\Sigma(\alpha - \beta)(\alpha - \gamma) = 9a$.

4.2 Solving an equation when two or more of its roots are connected by certain relations

In this section, we learn about finding the roots of an equation, when the relations between the roots of the equation are given. Synthetic division is useful for this purpose. First we discuss about this method.

4.2.1 Synthetic Division

This method consists of two parts :

(i) Finding the quotient and the remainder, when

$$a_0x^n + a_1x^{n-1} + \cdots + a_n \ (a_0 \neq 0) \text{ is divided by } (x-a).$$

(ii) Finding the quotient and the remainder, when

$$a_0x^n + a_1x^{n-1} + \cdots + a_n \ (a_0 \neq 0) \text{ is divided by the polynomial } x^2 - px - q.$$

4.2.2 Method of finding the quotient and the remainder when $a_0x^n + a_1x^{n-1} + \cdots + a_n \ (a_0 \neq 0)$ is divided by $(x - a)$.

Let Q(x) be the quotient and R be the remainder, when

$$a_0x^n + a_1x^{n-1} + \cdots + a_n (a_0 \neq 0) \text{ is divided by } (x-a).$$

Suppose $Q(x) = b_0x^{n-1} + b_1x^{n-2} + \cdots + b_{n-1},\ b_0 \neq 0$.

Then $a_0x^n + a_1x^{n-1} + \cdots + a_n = (x-a)(b_0x^{n-1} + b_1x^{n-2} + \cdots + b_{n-1}) + \mathrm{R}$

i.e., $a_0x^n + a_1x^{n-1} + \cdots + a_n = b_0x^n + (b_1 - ab_0)x^{n-1} + (b_2 - ab_1)x^{n-2}$

$$+ \cdots + (b_{n-1} - ab_{n-2})x + (\mathrm{R} - ab_{n-1}).$$

On comparing the coefficients of like powers of x, we obtain

$a_0 = b_0$ i.e., $b_0 = a_0$

$a_1 = b_1 - b_0a$ i.e., $b_1 = a_1 + ab_0$

$a_2 = b_2 - b_1a$ i.e., $b_2 = a_2 + ab_1$

....

$a_{n-1} = b_{n-1} - ab_{n-2}$ i.e., $b_{n-1} = a_{n-1} + ab_{n-2}$

$a_n = \mathrm{R} - ab_{n-1}$ i.e., $\mathrm{R} = a_n + ab_{n-1}$

We can find the coefficients $b_0, b_1, ..., b_{n-1}$ of the quotient and the remainder R, by the following method, known as **Horner's method of synthetic division.**

$$\begin{array}{c|cccccccc}
a & a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\
 & 0 & ab_0 & ab_1 & ab_2 & \cdots & ab_{n-2} & ab_{n-1} \\
\hline
 & a_0 = b_0 & b_1 & b_2 & b_3 & \cdots & b_{n-1} & \multicolumn{1}{|c}{R}
\end{array}$$

Description of the above process

1. First write down the coefficients of $x^n, x^{n-1}, ..., x, x^0$ successively in a row [if any term with $x^k\ (0 \le k < n)$ is missing, take the coefficient of it as zero].
2. Draw a vertical line to the left of a_0 and write a to the left of the vertical line on the same horizontal level as that of a_0.
3. Underneath a_0 write 0 and draw a horizontal line below it. Below the horizontal line and below 0, write the sum a_0+0 as the first term of the 3rd row, which is equal to b_0. Now multiply b_0 with a and write this product below a_1 in the second row. The sum a_1+ab_0 is b_1. Write this in the third row next to b_0. Continue this process until all the terms of the second and the third rows are filled.
4. From the table, the quotient is $b_0x^{n-1} + b_1x^{n-2} + \cdots + b_{n-1}$ and the remainder is $R = a_n + ab_{n-1}$.

4.2.3 Note : If the divisor is $(x + a)$ then the above method can be used by replacing a with $-a$, in order to obtain the quotient and the remainder.

4.2.4 Note : If the divisor is $ax - b = a\left(x - \frac{b}{a}\right)$, then replace a by $\frac{b}{a}$ and repeat the above process to obtain the quotient Q(x) and the remainder R. In this case, to get the exact quotient and remainder divide Q(x) and R by a.

4.2.5 Example : Find the quotient and the remainder, using synthetic division,

when $x^4 - 6x^3 + 3x^2 + 26x - 24$ is divided by $x-4$.

By the method of synthetic division, we have

$$\begin{array}{c|ccccc}
4 & 1 & -6 & 3 & 26 & -24 \\
 & 0 & 4 & -8 & -20 & 24 \\
\hline
 & 1 & -2 & -5 & 6 & \multicolumn{1}{|c}{0}
\end{array}$$

Thus the quotient is $x^3 - 2x^2 - 5x + 6$ and the remainder is zero.

4.2.6 Example : Divide $3x^4 - x^3 + 2x^2 - 2x - 4$ by $(x+2)$ and find the quotient and the remainder.

Here $a = -2$. By the method of synthetic division, we have

-2	3	-1	2	-2	-4
	0	-6	14	-32	68
	3	-7	16	-34	$\boxed{64}$

Thus the quotient is $3x^3 - 7x^2 + 16x - 34$ and the remainder is 64.

4.2.7 To find the quotient and the remainder when $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ is divided by the polynomial $x^2 - px - q$.

In this case, the procedure is as follows :

	a_0	a_1	a_2	a_3	$\ldots$	a_{n-2}	a_{n-1}	a_n
p	0	pb_0	pb_1	pb_2	$\ldots$	pb_{n-3}	pb_{n-2}	0
q	0	0	qb_0	qb_1	$\ldots$	qb_{n-4}	qb_{n-3}	qb_{n-2}
	b_0	b_1	b_2	b_3	$\ldots$	b_{n-2}	R_1	R_2

Description of the process

First write down the coefficients of $x^n, x^{n-1}, x^{n-2}, \cdots, x^2, x, x^0$ successively in a row. Draw a vertical line to the left of a_0. Write down p, q as column figures to the left of the vertical line in the 2nd and 3rd rows respectively. These are the negatives of the coefficient of x and the constant term in the divisor. Draw a horizontal line below the third row.

Put 0 in two rows underneath a_0. Write this sum $a_0 + 0 + 0$ as the first element in the fourth row. Let it be b_0. Next multiply b_0 with p and write this product underneath a_1 and write next column entry as zero. Add the three terms a_1, pb_0 and 0. Let this sum be b_1. Write this in 4th row underneath a_1. Multiply b_1 with p and b_0 with q and write these products underneath a_2. Let the sum of a_2, pb_1 and qb_0 be b_2. Continue this process until the terms under a_{n-1} are obtained. Name the entry in the 4th row under a_{n-1} as R_1, instead of naming it as b_{n-1}. Below a_n put 0 and qb_{n-2} in the second and the third rows respectively. Let the sum of a_n, 0 and qb_{n-2} be R_2. Write it in the 4th row below a_n.

Let $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$

and $Q(x) = b_0x^{n-2} + b_1x^{n-3} + \cdots + b_{n-2}$.

Then $(x^2 - px - q)Q(x)$

$$= b_0x^n + (b_1 - pb_0)\, x^{n-1} + \sum_{k=2}^{n-2} (b_k - pb_{k-1} - qb_{k-2})\, x^{n-k} - (pb_{n-2} + qp_{n-3})\, x - qp_{n-2}$$

$$= \sum_{k=0}^{n-2} a_k\, x^{n-k} + (a_{n-1} - \mathrm{R}_1)x + a_n - \mathrm{R}_2.$$

$$= p(x) - \mathrm{R}_1\, x - \mathrm{R}_2.$$

Therefore $p(x) = (x^2 - px - q)\ \mathrm{Q}(x) + \mathrm{R}_1 x + \mathrm{R}_2.$

Hence Q (x) is the quotient and $\mathrm{R}_1 x + \mathrm{R}_2$ is the remainder, when we divide $p(x)$ with $\mathrm{x}^2 - px - q$.

4.2.8 Example : Find the quotient and the remainder when

$2x^5 - 3x^4 + 5x^3 - 3x^2 + 7x - 9$ is divided by $x^2 - x - 3.$

By synthetic division we have

	2	−3	5	−3	7	−9
1	0	2	−1	10	4	0
3	0	0	6	−3	30	12
	2	−1	10	4	41	3

Thus $2x^3 - x^2 + 10x + 4$ is the quotient and $41x + 3$ is the remainder.

4.2.9 Example : Find the quotient and the remainder when

$x^4 - 11x^3 + 44x^2 - 76x + 48$ is divided by $x^2 - 7x + 12$.

By synthetic division we have

	1	−11	44	−76	48
7	0	7	−28	28	0
−12	0	0	−12	48	−48
	1	−4	4	0	0

Thus the required quotient is $x^2 - 4x + 4$ and the remainder is zero.

4.2.10 Trial and error method

To find a root of $f(x) = 0$, we have to find out a value α of x such that $f(\alpha) = 0$. Some times we can do this by inspection. This method is known as trial and error method.

For example, if we are able to find a root of a cubic equation by this method, then it is easy to solve the cubic equation. If $f(a).f(b) < 0$ [i.e., $f(a) > 0, f(b) < 0$ *or* $f(a) < 0, f(b) > 0$], for two real numbers a and b, then a root lies between a and b. This result is useful in searching for a root of the equation $f(x) = 0$. For example, if $f(2)f(6) < 0$, then there is a root of $f(x) = 0$ between 2 and 6.

4.2.11 Note : When a root $'a'$ of the equation $f(x) = 0$ is known or obtained by trial and error method, then $(x-a)$ is a factor of the polynomial $f(x)$ and by dividing $f(x)$ by this factor, we can reduce the degree of the equation by one and solve the resultant equation for the remaining roots.

4.2.12 Note : When a relation between the roots of the equation $f(x) = 0$ is given, using this information we can easily factorise the equation and thereby solve the equation very easily.

Now we illustrate all the above methods in the following solved problems.

4.2.13 Solved Problems

1. Problem : *Solve* $x^3 - 3x^2 - 16x + 48 = 0$.

Solution : Let $f(x) = x^3 - 3x^2 - 16x + 48$.

By inspection we see that $f(3) = 27 - 27 - 48 + 48 = 0$.

Hence 3 is a root of $f(x) = 0$.

Now we divide $f(x)$ by $(x-3)$, using synthetic division.

$$\begin{array}{r|rrrr} 3 & 1 & -3 & -16 & 48 \\ & 0 & 3 & 0 & -48 \\ \hline & 1 & 0 & -16 & \boxed{0} \end{array}$$

Thus the quotient is $(x^2 - 16)$ and the remainder is 0.

Therefore $f(x) = (x-3)(x^2-16) = (x-3)(x-4)(x+4)$.

Hence 3, -4, 4 are the roots of the given equation.

2. Problem : *Find the roots of* $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$.

Solution : Let $f(x) = x^4 - 16x^3 + 86x^2 - 176x + 105$.

By inspection we see that, $f(1) = 1 - 16 + 86 - 176 + 105 = 0$.

Hence 1 is a root of $f(x) = 0$.

Now we divide $f(x)$ by $(x-1)$, using synthetic division.

1	1	-16	86	-176	105
	0	1	-15	71	-105
	1	-15	71	-105	0

Therefore $x^4 - 16x^3 + 86x^2 - 176x + 105 = (x-1)(x^3 - 15x^2 + 71x - 105)$ ... (1)

Let $g(x) = x^3 - 15x^2 + 71x - 105$.

By inspection $g(3) = 0$. Hence 3 is a root of $g(x) = 0$.

Now we divide $g(x)$ by $(x-3)$, using synthetic division.

3	1	-15	71	-105
	0	3	-36	105
	1	-12	35	0

Therefore $g(x) = (x-3)(x^2 - 12x + 35)$. ... (2)

From (1) and (2)

$$f(x) = (x-1)\,(x-3)(x^2 - 12x + 35)$$

$$= (x-1)\,(x-3)(x-5)\,(x-7).$$

Hence 1, 3, 5 and 7 are the roots of the given equation.

Now we solve equations when a relation between some of the roots is given.

3. Problem : *Solve* $x^3 - 7x^2 + 36 = 0$, *given one root being twice the other.*

Solution : Let α, β, γ be the three roots of the given equation and $\beta = 2\alpha$.

Now we have $\alpha + \beta + \gamma = 7$

$$\alpha\beta + \beta\gamma + \gamma\alpha = 0$$

$$\alpha\beta\gamma = -36$$

On substituting $\beta = 2\alpha$ in the above equations, we obtain

$$3\alpha + \gamma = 7 \qquad ...(1)$$

$$2\alpha^2 + 3\alpha\gamma = 0 \qquad ...(2)$$

$$2\alpha^2\gamma = -36 \qquad ...(3)$$

On eliminating γ from (1) and (2), we have

$$2\alpha^2 + 3\alpha(7-3\alpha) = 0$$

i.e., $\alpha^2 - 3\alpha = 0$ or $\alpha(\alpha-3) = 0$

Therefore $\alpha = 0$ or $\alpha = 3$.

Since $\alpha = 0$ does not satisfy the given equation, we ignore this value.

Therefore $\alpha = 3$ is a root of the given equation.

So, $\beta = 6$ (since $\beta = 2\alpha$) and $\gamma = -2$.

Hence 3, 6, -2 are the roots of the given equation.

4. Problem : *Given that 2 is a root of* $x^3 - 6x^2 + 3x + 10 = 0$, *find the other roots.*

Solution : Let $f(x) = x^3 - 6x^2 + 3x + 10$.

Since 2 is a root of $f(x) = 0$, we divide $f(x)$ by $(x-2)$.

$$\begin{array}{c|cccc} 2 & 1 & -6 & 3 & 10 \\ & 0 & 2 & -8 & -10 \\ \hline & 1 & -4 & -5 & \boxed{0} \end{array}$$

Therefore $x^3 - 6x^2 + 3x + 10 = (x-2)(x^2 - 4x - 5)$

$$= (x-2)(x+1)(x-5).$$

Thus -1, 2 and 5 are the roots of the given equation.

5. Problem : *Given that two roots of* $4x^3 + 20x^2 - 23x + 6 = 0$ *are equal, find all the roots of the given equation.*

Solution : Let α, β and γ be the roots of $4x^3 + 20x^2 - 23x + 6 = 0$ ...(1)

Given that two roots of (1) are equal. So, let $\beta = \alpha$.

Since α, β, γ are the roots of (1), we have

$$\alpha + \beta + \gamma = -\frac{20}{4}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{23}{4}$$

$$\alpha\beta\gamma = -\frac{6}{4}$$

On substituting $\beta = \alpha$ in the above equations, we get

$$2\alpha + \gamma = -5 \quad \text{... (2)}$$

$$\alpha^2 + 2\alpha\gamma = -\frac{23}{4} \quad \text{... (3)}$$

$$\alpha^2\gamma = -\frac{3}{2} \quad \text{... (4)}$$

From (2) and (3), we get

$$\alpha^2 + 2\alpha(-5 - 2\alpha) = -\frac{23}{4}$$

i.e., $12\alpha^2 + 40\alpha - 23 = 0$

i.e., $(2\alpha - 1)(6\alpha + 23) = 0$

Therefore $\alpha = \frac{1}{2}$ or $\alpha = -\frac{23}{6}$.

On verification we get that $\alpha = \frac{1}{2}$ is a root of (1).

On substituting this value in (2), we get $\gamma = -6$.

Therefore $\frac{1}{2}, \frac{1}{2}, -6$ are the roots of (1).

6. Problem : *Given that the sum of two roots of* $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ ... (1) *is zero. Find the roots of the equation.*

Solution : Let α, β, γ and δ be the roots of (1), and $\alpha + \beta = 0$ ... (2)

From the relation between the coefficients and the roots, we have

$$\alpha + \beta + \gamma + \delta = 2, \text{ so } \gamma + \delta = 2 \quad \text{... (3)}$$

Therefore the quadratic equation having roots α and β is $x^2 - 0.x + \alpha\beta = 0$.

The quadratic equation having roots γ and δ is $x^2 - 2x + \gamma\delta = 0$

Let $\alpha\beta = p$, $\gamma\delta = q$.

Then $x^4 - 2x^3 + 4x^2 + 6x - 21 = (x^2 - 0.x + p)(x^2 - 2x + q)$

$$= (x^2 + p)(x^2 - 2x + q)$$

$$= x^4 - 2x^3 + (p + q)x^2 - 2px + pq.$$

On comparing the coefficients of x^2 and x on both sides, we obtain

$$p + q = 4 \text{ and } -2p = 6.$$

From the above equations we get $p = -3$ and $q = 7$.

Thus $x^4 - 2x^3 + 4x^2 + 6x - 21 = (x^2 - 3)(x^2 - 2x + 7)$.

The roots of $x^2-3=0$ are $\sqrt{3}, -\sqrt{3}$ and the roots of $x^2-2x+7=0$ are $1+i\sqrt{6}, 1-i\sqrt{6}$.

Hence $\sqrt{3}, -\sqrt{3}, 1+i\sqrt{6}, 1-i\sqrt{6}$ are the roots of (1).

7. Problem : *Solve* $4x^3-24x^2+23x+18=0$, *given that the roots of this equation are in arithmetic progression.*

Solution : Let $a-d,\ a,\ a+d$ be the roots of the given equation. These are in A.P.

From the relation between the coefficients and the roots, we have

$$(a-d)+a+(a+d)=\frac{-(-24)}{4}$$

$$(a-d)a+(a+d)a+(a-d)(a+d)=\frac{23}{4}$$

and $(a-d)a(a+d)=\dfrac{-18}{4}$

On simplifying these equations, we have

$$a = 2 \qquad \text{... (1)}$$

$$3a^2-d^2 = \frac{23}{4} \qquad \text{... (2)}$$

$$a^3-ad^2 = -\frac{9}{2} \qquad \text{...(3)}$$

From (1) and (2) we get $12-d^2=\dfrac{23}{4}$. Therefore $d=\pm\dfrac{5}{2}$.

Hence the roots of the given equation are $-\dfrac{1}{2}, 2$ and $\dfrac{9}{2}$.

8. Problem : *Solve* $x^3-7x^2+14x-8=0$, *given that the roots are in geometric progression.*

Solution : Let $\dfrac{a}{r},\ a,\ ar$ be the roots of the given equation. Then

$$\frac{a}{r}+a+ar=7 \qquad \text{...(1)}$$

$$\frac{a}{r}a+a.ar+\frac{a}{r}.a\ r=14 \qquad \text{...(2)}$$

$$\frac{a}{r}.a.\ ar=8 \qquad \text{...(3)}$$

Hence $a=2$. On substituting $a=2$ in (1), we obtain

$$\frac{2}{r}+2+2r=7$$

i.e., $2r^2-5r+2=0$

i.e., $(r-2)(2r-1)=0$

Therefore $r=2$ or $r=\dfrac{1}{2}$.

Hence the roots of the given equation are 1,2 and 4.

9. Problem : *Solve* $x^4-5x^3+5x^2+5x-6=0$, *given that the product of two of its roots is 3.*

Solution : Let $\alpha,\beta,\gamma,\delta$ be the roots of the given equation.

Then $\alpha+\beta+\gamma+\delta=5$ ...(1)

$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=5$ ...(2)

$\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta=-5$ ... (3)

$\alpha\beta\gamma\delta=-6$ ...(4)

Given that the product of two roots of the given equation is 3.

Suppose that

$\gamma\delta=3$ ...(5)

From (4) and (5), we get

$\alpha\beta=-2$ ... (6)

From (3), (5) and (6) we get

$-2\gamma+3\beta+3\alpha-2\delta=-5$

i.e., $3(\alpha+\beta)-2(\gamma+\delta)=-5$ ... (7)

From (1) and (7), we obtain

$\alpha+\beta=1$ ... (8)

$\gamma+\delta=4$ ... (9)

Now $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=1-4(-2)=9$.

Therefore $(\alpha-\beta)=\pm 3$. ...(10)

On solving (8) and (10), we get $\alpha=2$ and $\beta=-1$ or $\alpha=-1$ and $\beta=2$.

Now $(\gamma-\delta)^2=(\gamma+\delta)^2-4\gamma\delta=16-4(3)=4$.

Therefore $\gamma-\delta=\pm 2$. ... (11)

On solving (9) and (11), we get $\gamma=3$ and $\delta=1$ or $\gamma=1$ and $\delta=3$.

Hence the roots of the given equation are 2, –1, 3 and 1.

10. Problem : *Solve* $x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$, *given that it has two pairs of equal roots.*

Solution : Let the roots of the given equation be $\alpha, \alpha, \beta, \beta$.

Then $\quad 2\alpha + 2\beta = -4 \quad$ ... (1)

$\alpha^2\beta + \alpha^2\beta + \alpha\beta^2 + \alpha\beta^2 = 12$. ... (2)

From (1), we get

$\alpha + \beta = -2$. ... (3)

From (2), we get

$2\alpha\beta(\alpha + \beta) = 12$.

Hence, from (3), we have $\alpha\beta(-4) = 12$,

i.e., $\alpha\beta = -3$ ... (4)

Now, $\quad (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = 4 - 4(-3) = 16$.

Therefore $\quad \alpha - \beta = \pm 4$.

On solving $\quad \alpha + \beta = -2,\ \alpha - \beta = 4,\quad$ we get $\ \alpha = 1,\ \beta = -3$.

If we take $\quad \alpha - \beta = -4$, then we get $\alpha = -3$ and $\beta = 1$.

Therefore $1, 1, -3, -3$ are the roots of the given equation.

11. Problem : *Prove that the sum of any two of the roots of the equation* $x^4 + px^3 + qx^2 + rx + s = 0$ *is equal to the sum of the remaining two roots of the equation iff* $p^3 - 4pq + 8r = 0$.

Solution : Suppose that the sum of two of the roots of the given equation is equal to the sum of the remaining two roots.

Let $\alpha, \beta, \gamma, \delta$ be the roots of the given equation such that $\alpha + \beta = \gamma + \delta$.

Then $\quad x^4 + px^3 + qx^2 + rx + s = (x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$

$$= [x^2 - (\alpha + \beta)x + \alpha\beta][x^2 - (\gamma + \delta)x + \gamma\delta]$$

$$= [x^2 - (\alpha + \beta)x + \alpha\beta][x^2 - (\alpha + \beta)x + \gamma\delta]$$

Alternate Solution : Let $\alpha,\ \beta,\ \gamma,\ \delta$ be the roots of the given equation.

Then $x^4 + px^3 + qx^2 + rx + s$

$= (x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$

$= [x^2 - (\alpha + \beta)x - \alpha\beta][x^2 - (\gamma + \delta)x + \gamma\delta]$

Now $\quad \alpha + \beta = \gamma + \delta \Leftrightarrow x^4 + px^3 + qx^2 + \gamma x + s = [x^2 - (\alpha + \beta)x + \alpha\beta][x^2 - (\alpha + \beta) + \gamma\delta]$

$= (x^2 + bx + c)(x^2 + bx + d) \qquad$ for $-(\alpha + \beta) = b,\ \alpha\beta = c,\ \gamma\delta = d$.

$\Leftrightarrow 2b = p, b^2 + c + d = q, b(c+d) = r, cd = s$

$\Leftrightarrow p^3 - 4pq + 8r = 0$

Let $b = -(\alpha+\beta), c = \alpha\beta, d = \gamma\delta$. Then

$$x^4 + px^3 + qx^2 + rx + s = (x^2+bx+c)(x^2+bx+d)$$
$$= (x^2+bx)^2 + (c+d)(x^2+bx) + cd$$
$$= x^4 + 2bx^3 + (b^2+c+d)x^2 + b(c+d)x + cd$$

Hence

$2b = p$ ...(1)

$b^2 + c + d = q$ ...(2)

$b(c+d) = r$ ...(3)

$cd = s$ ...(4)

From these equations, we have

$$b = \frac{p}{2}, \; c+d = q - b^2 = q - \left(\frac{p}{2}\right)^2$$

and $\quad \dfrac{p}{2}\left(q - \dfrac{p^2}{4}\right) = b(c+d) = r$.

Now $\quad \dfrac{p}{2}\left(q - \dfrac{p^2}{4}\right) = r \Leftrightarrow p^3 - 4pq + 8r = 0$.

Conversely suppose that $p^3 - 4pq + 8r = 0$.

Then $\dfrac{p}{2}\left(q - \dfrac{p^2}{4}\right) = r$. ...(5)

Then choose c and d such that $c + d = q - \dfrac{p^2}{4}$ and $cd = s$.

Take $b = \dfrac{p}{2}$.

Then equations (1), (2) and (4) are satisfied. In view of (5), equation (3) is also satisfied.

Hence $(x^2+bx+c)(x^2+bx+d) = x^4 + 2bx^3 + (b^2+c+d)x^2 + b(c+d)x + cd$
$$= x^4 + px^3 + qx^2 + rx + s$$

Hence the roots of the given equation are α_1, β_1, γ_1 and δ_1, where α_1 and β_1 are the roots of the equations $x^2 + bx + c = 0$ and γ_1 and δ_1 are those of the equation $x^2 + bx + d = 0$.

We have $\alpha_1 + \beta_1 = -b = \gamma_1 + \delta_1$.

4.2.14 Definition

Let $f(x)$ be a polynomial of degree $n > 0$. Let $\alpha_1, \alpha_2, ..., \alpha_n$ be the roots of $f(x) = 0$ so that $f(x) = a_0(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$. A complex number α is said to be a root of $f(x) = 0$ of multiplicity m, if $\alpha = \alpha_k$ for exactly m values of k among 1, 2, ..., n. Roots of multiplicity $m > 1$ are called ***multiple roots*** *or* ***repeated roots****. Roots of multiplicity 1 are called* ***simple roots****.*

We note that if $f(x)$ is a polynomial of degree $n > 0$, then $\alpha \in \mathbf{C}$ is a root of $f(x) = 0$ of multiplicity $m > 0$ iff $f(x) = (x-\alpha)^m Q(x)$ for some polynomial $Q(x)$ of degree $n-m$ with $Q(\alpha) \neq 0$.

4.2.15 Theorem : *Let $f(x)$ be a polynomial of degree $n > 0$. Let α be a root of the equation $f(x) = 0$ of multiplicity m. If $m > 1$, then α is a root of the equation* $f'(x) = 0$ of multiplicity $m-1$. If $m = 1$, then $f'(\alpha) \neq 0$.

Proof : Since α is a root of $f(x) = 0$ of multiplicity m, we have

$$f(x) = (x-\alpha)^m g(x) \qquad ... (1)$$

for some polynomial $g(x)$ of degree $n-m$ with $g(\alpha) \neq 0$.

On differentiating (1) *w. r. t. x*, we get

$$f'(x) = (x-\alpha)^m g'(x) + m(x-\alpha)^{m-1} g(x)$$

$$= (x-\alpha)^{m-1}\{m\,g(x) + (x-\alpha)\,g'(x)\}$$

$$= (x-\alpha)^{m-1} h(x).$$

where $h(x) = (x-\alpha)g'(x) + m\,g(x)$. Here $h(\alpha) \neq 0$, since $g(\alpha) \neq 0$.

Therefore α is not a root of $h(x) = 0$.

Thus α is a root of the equation $f'(x) = 0$ of multiplicity $m-1$, if $m > 1$.

If $m = 1$, then $f'(\alpha) = h(\alpha) \neq 0$.

4.2.16 Corollary : Let $f(x)$ be a polynomial of degree $n > 0$. If α is a root of $f(x) = 0$ of multiplicity m, then α is a root of $f^{(k)}(x) = 0$ of multiplicity $m-k\,(k = 1, 2, ..., m-1)$.

4.2.17 Corollary : Let $f(x)$ be a polynomial of degree $n > 0$. Then α is a root of the equation $f(x) = 0$ of multiplicity m iff $f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0$ and $f^{(m)}(\alpha) \neq 0$.

4.2.18 Note : From the above theorem, for a polynomial $f(x)$ we observe the following :

1. If $f(x)$ contains a factor of the form $(x-\alpha)^m$ for some $\alpha \in \mathbf{C}$ and some positive integer $m > 1$, then $f'(x)$ contains the factor $(x-\alpha)^{m-1}$. Thus $f(x)$ and $f'(x)$ have a common factor $(x-\alpha)^{m-1}$.

2. If $f(x)$ and $f'(x)$ have no common factor ($f(x)$ is a polynomial of degree > 0), then the equation $f(x)=0$ has no multiple roots.

4.2.19 Procedure to find multiple roots

Let $f(x)$ be a polynomial. First we find $f'(x)$ and then find the H.C.F. of $f(x)$ and $f'(x)$. Now we note that, if α is a root of the H.C.F. of multiplicity *k,* then α is a multiple root of order $(k+1)$ of $f(x)=0$.

4.2.20 Example : We find the roots of $27x^3 - 36x - 16 = 0$ given that there is a multiple root.

Let $f(x) = 27x^3 - 36x - 16$.

Then $f'(x) = 81x^2 - 36 = 9(9x^2-4) = 9(3x+2)(3x-2)$.

We have $f\left(-\dfrac{2}{3}\right) = 0$ and $f\left(\dfrac{2}{3}\right) \neq 0$.

Hence $3x+2$ is the only common factor of $f(x)$ and $f'(x)$.

Therefore $-\dfrac{2}{3}$ is a repeated root of order 2 of $f(x)=0$.

On dividing $27x^3 - 36x - 16$ with $(3x+2)^2$, we obtain $(3x-4)$ as the quotient and remainder zero.

So $27x^3 - 36x - 16 = (3x+2)^2\,(3x-4)$

Hence the roots of the given equation are $-\dfrac{2}{3}, -\dfrac{2}{3}$ and $\dfrac{4}{3}$.

4.2.21 Example : Prove that $x^5 - 5x^3 + 5x^2 - 1 = 0$ has three equal roots and find this root.

Solution: Let $f(x) = x^5 - 5x^3 + 5x^2 - 1$

$$f'(x) = 5x^4 - 15x^2 + 10x$$

$$f''(x) = 20x^3 - 30x + 10$$

Sum of the coefficients of $f(x)$ is 0

$\Rightarrow$ 1 is a root of $f(x)=0$

$\Rightarrow$ $(x-1)$ is a factor of $f(x)$.

For the same reason $x-1$ is also a factor of $f'(x)$ and $f''(x)$.

Thus $f(x)=0$ has three equal roots and it is 1.

Exercise 4(b)

I. 1. Solve $x^3-3x^2-16x+48=0$, given that the sum of two roots is zero.

2. Find the condition that $x^3-px^2+qx-r=0$ may have the sum of two of its roots as zero.

3. Given that the roots of $x^3+3px^2+3qx+r=0$ are in

(i) A.P., show that $2p^3-3qp+r=0$

(ii) G.P., show that $p^3r=q^3$

(iii) H.P., show that $2q^3=r(3pq-r)$

4. Find the condition that $x^3-px^2+qx-r=0$ may have the roots in G.P.

II. 1. Solve $9x^3-15x^2+7x-1=0$, given that two of its roots are equal.

2. Given that one root of $2x^3+3x^2-8x+3=0$ is double the other root, find the roots of the equation.

3. Solve $x^3-9x^2+14x+24=0$, given that two of the roots are in the ratio 3:2.

4. Solve the following equations, given that the roots of each are in A.P.

(i) $8x^3-36x^2-18x+81=0$ (ii) $x^3-3x^2-6x+8=0$

5. Solve the following equations, given that the roots of each are in G.P.

(i) $3x^3-26x^2+52x-24=0$ (ii) $54x^3-39x^2-26x+16=0$

6. Solve the following equations, given that the roots of each are in H.P.

(i) $6x^3-11x^2+6x-1=0$ (ii) $15x^3-23x^2+9x-1=0$

7. Solve the following equations, given that they have multiple roots.

(i) $x^4-6x^3+13x^2-24x+36=0$ (ii) $3x^4+16x^3+24x^2-16=0$

III. 1. Solve $x^4 + x^3 - 16x^2 - 4x + 48 = 0$, given that the product of two of the roots is 6.

2. Solve $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$ given that two of its roots have the same absolute value, but have opposite sign.

3. Solve $18x^3 + 81x^2 + 121x + 60 = 0$ given that one root is equal to half the sum of the remaining roots.

4. Find the condition in order that the equation $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$ may have two pairs of equal roots.

5. (i) Show that $x^5 - 5x^3 + 5x^2 - 1 = 0$ has three equal roots and find this root. (ii) Find the repeated roots of $x^5 - 3x^4 - 5x^3 + 27x^2 - 32x + 12 = 0$.

6. Solve the equation $8x^3 - 20x^2 + 6x + 9 = 0$ given that the equation has a multiple root.

4.3 Equations with real coefficients - occurrence of complex roots in conjugate pairs and its consequences

Now we prove that, if a root of a polynomial equation with real coefficients is a complex number, then its conjugate is also a root of it. Similarly we prove that, in an equation with rational coefficients, irrational roots occur in pairs.

4.3.1 Lemma

Let $f(x)$ be a polynomial with real coefficients.

Let $\alpha \in \mathbf{C}$. Then $f(\overline{\alpha}) = \overline{f(\alpha)}$.

Proof :

Let $$f(x) = \sum_{k=0}^{n} a_k x^{n-k}$$

Then $$f(\overline{\alpha}) = \sum_{k=0}^{n} a_k (\overline{\alpha})^{n-k} = \sum_{k=0}^{n} a_k \left(\overline{\alpha^{n-k}}\right)$$

$$= \sum_{k=0}^{n} \left(\overline{a_k \alpha^{n-k}}\right) \text{ (since } a_k \text{ is real for all } k)$$

$$= \left[\overline{\sum_{k=0}^{n} a_k \alpha^{n-k}}\right] = \overline{f(\alpha)}.$$

4.3.2 Theorem : *Let f(x) be a polynomial of degree n > 0, with real coefficients. Let α be a root of the equation f(x) = 0 of multiplicity m. Then $\overline{\alpha}$ is a root of f(x) = 0 of multiplicity m.*

Proof : Since α is a root of $f(x)=0$ of multiplicity m, we have

$$f^{(k)}(\alpha)=0 \text{ for } k=0,1,\ldots,\ m-1 \text{ and } f^{(m)}(\alpha)\neq 0.$$

Since $f(x)$ is a polynomial with real coefficients, so is $f^{(k)}(x)$ for $k=0,1,\ldots,\ m-1$

and hence $f^{(k)}(\overline{\alpha})=\left[\overline{f^{(k)}(\alpha)}\right],\ \forall\ k=0,1,2,\ldots,m-1.$

Therefore $f^{(k)}(\overline{\alpha})=0$ for $k=0,1,\ldots,m-1$ and $f^{(m)}(\overline{\alpha})\neq 0$.

Hence $\overline{\alpha}$ is a root of the equation $f(x)=0$ of multiplicity m.

4.3.3 Note: In a polynomial equation with real coefficients, complex roots occur in conjugate pairs with the same multiplicity.

4.3.4 Theorem : *Let f(x) be a polynomial of degree n > 0, with real coefficients and leading coefficient a_0. Suppose that none of the roots of the equation f(x) = 0 is real. Then $\dfrac{f(\alpha)}{a_0}>0$ for all real α* and *n is an even integer.*

Proof : Since the equation $f(x)=0$ has no real roots, each root of it is a non-real complex number.

Since the coefficients of $f(x)$ are real, the non-real complex roots of $f(x)=0$ occur in conjugate pairs with the same multiplicity.

Let $\gamma_1,\overline{\gamma_1},\ \gamma_2,\overline{\gamma_2},\ldots,\ \gamma_m,\ \overline{\gamma_m}$ be all the roots of the equation $f(x)=0$.

$$\text{Hence } f(x)=a_0\prod_{j=1}^{m}(x-\gamma_j)(x-\overline{\gamma_j}) \qquad \ldots(1)$$

Let $\gamma_j=\alpha_j+i\beta_j$, where $\alpha_j,\ \beta_j$ are real.

Then $\beta_j\neq 0$ and $(x-\gamma_j)(x-\overline{\gamma_j})=(x-\alpha_j-i\beta_j)(x-\alpha_j+i\beta_j)=(x-\alpha_j)^2+\beta_j^2$

$$\text{Hence } f(x)=a_0\prod_{j=1}^{m}\left[\left(x-\alpha_j\right)^2+\beta_j^{\,2}\right]$$

We have $\left(\alpha-\alpha_j\right)^2+\beta_j^{\,2}\geq\beta_j^{\,2}>0,\forall\ \alpha\in\mathbf{R}$ and for all $j=1,2,\ldots,m$.

$$\text{Hence } \frac{f(\alpha)}{a_0}\geq\prod_{j=1}^{m}\beta_j^{\,2}>0\ \forall\,\alpha\in\mathbf{R}.$$

On comparing the degrees of both sides of (1) we get that $n=2m$ and hence n is an even integer.

4.3.5 Note : Let $f(x)$ be a polynomial of degree $n > 0$, with real coefficients. Let a_0 be the leading coefficient of $f(x)$.

1. If the equation $f(x) = 0$ has no real roots, then n is even and $f(\alpha)$ and a_0 have the same sign for all real α.
2. If n is odd, then the equation $f(x) = 0$ has atleast one real root.

4.3.6 Example: Form the cubic equation with real coefficients, two roots of which are $1, 3-\sqrt{-2}$.

Since $3-\sqrt{-2}$ is a root of the given equation, $3+\sqrt{-2}$ is also a root.

So we form an equation whose roots are $1, 3+\sqrt{-2}, 3-\sqrt{-2}$.

Hence the required equation is

$$(x-1)(x-3-\sqrt{-2})(x-3+\sqrt{-2}) = 0$$

i.e., $(x-1)\left\{(x-3)^2+2\right\} = 0$

i.e., $(x-1)(x^2-6x+11) = 0$

i.e., $x^3-7x^2+17x-11 = 0.$

4.3.7 Example: Solve the equation $x^4+4x^3+5x^2+2x-2=0$, one root of which is $-1+\sqrt{-1}$.

Let $f(x) = x^4+4x^3+5x^2+2x-2$.

In a polynomial equation with real coefficients, complex roots occur in conjugate pairs.

Hence $-1-\sqrt{-1}$ is also a root of the given equation.

Therefore $(x+1-\sqrt{-1})(x+1+\sqrt{-1})$ is a factor of $f(x)$.

Hence $f(x)$ is exactly divisible by $(x+1-\sqrt{-1})(x+1+\sqrt{-1})$,

i.e., $(x+1)^2+1$, i.e., x^2+2x+2.

On dividing $f(x)$ by x^2+2x+2, we get the quotient x^2+2x-1. Therefore

$$x^4+4x^3+5x^2+2x-2 = (x^2+2x+2)(x^2+2x-1).$$

Hence the other roots are obtained from $x^2+2x-1=0$, which are $-1\pm\sqrt{2}$.

Thus the roots of the given equation are $-1\pm\sqrt{-1}$ and $-1\pm\sqrt{2}$.

4.3.8 Example: Show that $\dfrac{a^2}{x-\alpha}+\dfrac{b^2}{x-\beta}+\dfrac{c^2}{x-\gamma}-x+\delta=0$

has only real roots, if $a, b, c, \alpha, \beta, \gamma, \delta$ are all real.

Let $f(x)=\dfrac{a^2}{x-\alpha}+\dfrac{b^2}{x-\beta}+\dfrac{c^2}{x-\gamma}-x+\delta$

Suppose that $f(x)=0$ has non-real complex root $p+iq$.

$$\text{Then}\quad \frac{a^2}{p+iq-\alpha}+\frac{b^2}{p+iq-\beta}+\frac{c^2}{p+iq-\gamma}-p-iq+\delta=0. \qquad \text{... (1)}$$

On taking the conjugate of (1), we get

$$\frac{a^2}{p-iq-\alpha}+\frac{b^2}{p-iq-\beta}+\frac{c^2}{p-iq-\gamma}-p+iq+\delta=0. \qquad \text{... (2)}$$

On subtracting (2) from (1), we get

$$-\frac{2a^2iq}{(p-\alpha)^2+q^2}-\frac{2b^2iq}{(p-\beta)^2+q^2}-\frac{2c^2iq}{(p-\gamma)^2+q^2}-2iq=0.$$

$$\text{i.e., } -2iq\left\{\frac{a^2}{(p-\alpha)^2+q^2}+\frac{b^2}{(p-\beta)^2+q^2}+\frac{c^2}{(p-\gamma)^2+q^2}+1\right\}=0$$

$\Rightarrow q=0$ ($\because$ the quantity inside the bracket is nonzero)

$\Rightarrow p+iq=p,$ a real number.

Hence all the roots of the given equation are real.

4.3.9 Theorem : *Let f(x) be a polynomial of degree n > 0 with rational coefficients. Let a and b be rational numbers,* $b>0$ and $\sqrt{b}$ *irrational. Then* $a+\sqrt{b}$ *is a root of f(x) = 0 if and only if so is* $a-\sqrt{b}$.

Proof : Without loss of of generality, we may assume that $a+\sqrt{b}$ is a root of $f(x)=0$. Then, $(x-a-\sqrt{b})$ is a factor of $f(x)$. Since the coefficients of $f(x)$ are rational numbers, $(x-a-\sqrt{b})$ is a factor of $f(x)$ and $\sqrt{b}$ is an irrational number, it follows that $n\neq 1$. Hence $n\geq 2$. We have

$$(x-a-\sqrt{b})\,(x-a+\sqrt{b})=(x-a)^2-b \qquad \text{... (1)}$$

When $f(x)$ is divided by $(x-a)^2-b$, let the quotient be $Q(x)$ and the remainder be R_1x+R_2.

Since the coefficients of $f(x)$ and a and b are rational numbers, it follows that the coefficients of $Q(x)$, R_1 and R_2 are all rational numbers.

$$\text{We have}\qquad f(x)=\left\{(x-a)^2-b\right\}Q(x)+R_1x+R_2 \qquad \text{... (2)}$$

On substituting $a+\sqrt{b}$ for x in (2), we get

$$f(a+\sqrt{b})=\left\{(a+\sqrt{b}-a)^2-b\right\}Q(a+\sqrt{b})+R_1(a+\sqrt{b})+R_2=R_1(a+\sqrt{b})+R_2,$$

Since $a+\sqrt{b}$ is a root of the equation $f(x)=0$, $f(a+\sqrt{b})=0$.

Therefore $R_1a+R_2+R_1\sqrt{b}=0$. ... (3)

Hence $-(R_1a+R_2)=R_1\sqrt{b}$.

If $R_1\neq 0$, then $\sqrt{b}=\dfrac{-(R_1a+R_2)}{R_1}$, which is rational, a contradiction.

Hence $R_1=0$ and from (3), it follows that $R_2=0$.

Therefore $a-\sqrt{b}$ is a root of $f(x)=0$.

Similarly we can prove that if $a-\sqrt{b}$ is a root of $f(x)=0$., then $a+\sqrt{b}$ is a root of $f(x)=0$.

4.3.10 Example: Solve the equation $x^4-5x^3+4x^2+8x-8=0$, given that one of the roots is $1-\sqrt{5}$.

Since the coefficients in the given equation are rational numbers and $1-\sqrt{5}$ is a root of it, $1+\sqrt{5}$ is also a root of it. The factors corresponding to these roots are $x-1+\sqrt{5}$ and $x-1-\sqrt{5}$. Then we have

$$(x-1+\sqrt{5})(x-1-\sqrt{5})=(x-1)^2-5=x^2-2x-4.$$

On dividing $x^4-5x^3+4x^2+8x-8$ by x^2-2x-4, we get the quotient x^2-3x+2.

Therefore $x^4-5x^3+4x^2+8x-8=(x^2-2x-4)\,(x^2-3x+2)$

$$=(x^2-2x-4)\,(x-1)(x-2).$$

Hence the roots of the given equation are $1\pm\sqrt{5}$, 1, 2.

4.3.11 Solved Problems

1. Problem : *Form the monic polynomial equation of degree 4 whose roots are* $4+\sqrt{3}$, $4-\sqrt{3}$, $2+i$ and $2-i$.

Solution : The required equation is

$$\left\{x-(4+\sqrt{3})\right\}\left\{x-(4-\sqrt{3})\right\}\{x-(2+i)\}\{x-(2-i)\}=0.$$

i.e., $(x^2-8x+13)(x^2-4x+5)=0$

i.e., $x^4-12x^3+50x^2-92x+65=0$.

2. Problem : *Solve* $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$, *given that one of its roots is* $2 + \sqrt{3}$.

Solution : Since $2 + \sqrt{3}$ is a root of the given equation, by Theorem 4.3.9, $2 - \sqrt{3}$ is also a root of it.

The quadratic factor corresponding to these two roots is $x^2 - 4x + 1$. On dividing $6x^4 - 13x^3 - 35x^2 - x + 3$ by $x^2 - 4x + 1$ (by synthetic division) we get the quotient $6x^2 + 11x + 3$.

Therefore $6x^4 - 13x^3 - 35x^2 - x + 3 = (x^2 - 4x + 1)(6x^2 + 11x + 3)$

Hence the other roots are obtained from $6x^2 + 11x + 3 = 0$.

On solving this equation, we get $x = -\frac{1}{3}$ or $-\frac{3}{2}$.

Thus the roots of the given equation are $-\frac{1}{3}, -\frac{3}{2}, 2 \pm \sqrt{3}$.

Exercise 4(c)

I. 1. Form the polynomial equation, whose roots are

(i) $2 + 3i,\ 2 - 3i, 1 + i, 1 - i$ (ii) $3, 2, 1 + i, 1 - i$

(iii) $1 + i, 1 - i,\ -1 + i,\ -1 - i$ (iv) $1 + i,\ 1 - i,\ 1 + i,\ 1 - i$

2. Form the polynomial equation of least degree with rational coefficients whose roots are

(i) $4\sqrt{3},\ 5 + 2i$ (ii) $1 + 5i, 5 - i$

(iii) $i - \sqrt{5}$ (iv) $-\sqrt{3} + i\sqrt{2}$

II. 1. Solve the equation $x^4 + 2x^3 - 5x^2 + 6x + 2 = 0$ given that $1 + i$ is one of its roots.

2. Solve the equation $3x^3 - 4x^2 + x + 88 = 0$ which has $2 - \sqrt{-7}$ as a root.

3. Solve $x^4 - 4x^2 + 8x + 35 = 0$, given that $2 + i\sqrt{3}$ is a root.

4. Solve the equation $x^4 + 6x^3 + 11x^2 - 10x + 2 = 0$, given that $2 + \sqrt{3}$ is a root of the equation.

5. Given that $-2 + \sqrt{-7}$ is a root of the equation $x^4 + 2x^2 - 16x + 77 = 0$, solve it completely.

6. Solve the equation $x^4 + 2x^3 - 16x^2 - 22x + 7 = 0$, given that $2 - \sqrt{3}$ is one of its roots.

7. Solve the equation $3x^5 - 4x^4 - 42x^3 + 56x^2 + 27x - 36 = 0$, given that $\sqrt{2} + \sqrt{5}$ is one of its roots.

8. Solve the equation $x^4 - 9x^3 + 27x^2 - 29x + 6 = 0$, given that one root is $2 - \sqrt{3}$.

9. Show that the equation

$$\frac{a^2}{x-a'} + \frac{b^2}{x-b'} + \frac{c^2}{x-c'} + \ldots + \frac{k^2}{x-k'} = x - m,$$

where $a, b, c, \ldots, k, \; m, a', b', \ldots, k'$ are all real numbers, can not have a non-real root.

4.4 Transformation of equations - Reciprocal equations

If an equation is given, it is possible to transform this equation into another equation whose roots have some relation with the roots of the original equation (even if the actual roots of the original equation are not known). Such a transformation often helps us to solve equations easily or to discuss the nature of the roots of the equations. We explain here some important elementary transformations of equations.

Throughout this section, unless otherwise stated, $f(x)$ stands for a polynomial.

4.4.1 Theorem (Roots with change of sign) : *If* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are the roots of* $f(x) = 0$, *then* $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$ *are the roots of* $f(-x) = 0$.

Proof: Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $f(x) = 0$.

Then $f(x) = a_0 (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$, where a_0 is the leading coefficient of $f(x)$.

Hence $f(-x) = a_0 (-x - \alpha_1)(-x - \alpha_2) \ldots (-x - \alpha_n)$

$$= (-1)^n a_0 (x + \alpha_1)(x + \alpha_2) \ldots (x + \alpha_n)$$

Therefore $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$ are the roots of $f(-x) = 0$

4.4.2 Note : Let $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ with $a_0 \neq 0$.

$$\begin{aligned} \text{Then} \quad f(-x) &= a_0(-x)^n + a_1(-x)^{n-1} + a_2(-x)^{n-2} + \cdots + a_{n-1}(-x) + a_n \\ &= (-1)^n [a_0 x^n - a_1 x^{n-1} + a_2 x^{n-2} - a_3 x^{n-3} + \cdots \\ &\qquad + (-1)^{n-1} a_{n-1} x + (-1)^n a_n]. \end{aligned}$$

Hence from the above theorem it follows that if we change the sign of alternate terms of the given polynomial equation beginning with the second, we obtain a polynomial equation whose roots are the negatives of those of the given equation and which have the same multiplicities as that of the corresponding ones of the given equation. In particular, for any $\alpha \in \mathbf{C}, -\alpha$ is a root of $f(-x) = 0$ of multiplicity m iff α is a root of $f(x) = 0$ of multiplicity m.

4.4.3 Theorem (Roots multiplied by a given number)

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are the roots of* $f(x) = 0$, *then for any non-zero complex number* k, *the roots of* $f\left(\frac{x}{k}\right) = 0$ are $k\alpha_1, k\alpha_2, \ldots, k\alpha_n$.

Proof : Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $f(x) = 0$, so that

$$f(x) = a_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n),$$

where a_0 is the leading coefficient of $f(x)$.

Then

$$f\left(\frac{x}{k}\right) = a_0\left(\frac{x}{k}-\alpha_1\right)\left(\frac{x}{k}-\alpha_2\right)\ldots\left(\frac{x}{k}-\alpha_n\right)$$

$$= \frac{a_0}{k^n}(x-k\alpha_1)(x-k\alpha_2)\ldots(x-k\alpha_n).$$

Hence the roots of $f\left(\frac{x}{k}\right) = 0$ are $k\alpha_1, k\alpha_2, \ldots, k\alpha_n$.

4.4.4 Note : Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$.

Let k be a non-zero complex number. Then

$$f\left(\frac{x}{k}\right) = a_0\left(\frac{x}{k}\right)^n + a_1\left(\frac{x}{k}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{x}{k}\right) + a_n$$

$$= \frac{1}{k^n}\left[a_0x^n + ka_1x^{n-1} + k^2a_2x^{n-2} + \cdots + k^{n-1}a_{n-1}x + k^na_n\right].$$

Hence from the above theorem it follows that, if we multiply the successive terms of a given polynomial equation of degree $n > 0$ beginning with the second by $k, k^2, \ldots, k^n$ respectively, then we obtain a polynomial equation whose roots are k times those of the given equation and which have the same multiplicities as those of the corresponding ones of the given equation.

4.4.5 Note : Theorem 4.4.1 is a particular case of Theorem 4.4.3 for $k = -1$.

4.4.6 Theorem : *If* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are the roots of the polynomial equation* $f(x) = 0$, *then* $\alpha_1 - h, \alpha_2 - h, \ldots, \alpha_n - h$ *are the roots of the equation* $f(x+h) = 0$.

Proof : Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of $f(x) = 0$.

Then, we have $f(x) = a_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$, where a_0 is the leading coefficient of the polynomial $f(x)$.

Hence $f(x+h) = a_0(x+h-\alpha_1)(x+h-\alpha_2)\ldots(x+h-\alpha_n)$

$$= a_0\left(x-(\alpha_1-h)\right)\left(x-(\alpha_2-h)\right)\ldots\left(x-(\alpha_n-h)\right).$$

Therefore $\alpha_1-h, \alpha_2-h, \ldots, \alpha_n-h$ are the roots of the equation $f(x+h)=0$.

4.4.7 Corollary : If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $f(x)=0$, then $\alpha_1+h, \alpha_2+h, \ldots, \alpha_n+h$ are the roots of the equation $f(x-h)=0$.

4.4.8 Note : Let $f(x)$ be a polynomial of degree $n > 0$. Let $h \in \mathbf{C}$. Then $f(x+h)$ is evidently a polynomial of degree n. Hence there exist unique constants $A_0, A_1, \ldots, A_n$ such that

$$f(x+h) = A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n.$$

On replacing x with $x-h$ in the above equation, we obtain

$$f(x) = A_0(x-h)^n + A_1(x-h)^{n-1} + \cdots + A_{n-1}(x-h) + A_n.$$

For $k = 0, 1, 2, \ldots, n$, let $Q_k(x) = A_0(x-h)^{n-k} + A_1(x-h)^{n-k-1} + \cdots + A_{n-k}$.

Then $Q_0(x) = f(x)$, $Q_n(x) = A_0$

and $Q_k(x) = (x-h)Q_{k+1}(x) + A_{n-k}$ $\quad (k = 0, 1, \ldots, n)$.

Thus A_{n-k} is the remainder and $Q_{k+1}(x)$ is the quotient that we obtain when we divide $Q_k(x)$ with $x-h$.

Since $Q_0(x) = f(x)$, beginning with $k=0$, by dividing $Q_k(x)$ with $x-h$, we can determine $Q_{k+1}(x)$ and A_{n-k} in a successive manner by using synthetic division. Thus we can determine the constants $A_n, A_{n-1}, \ldots, A_0$ in that order one after the other.

4.4.9 Theorem (Reciprocal roots) : *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the polynomial equation $f(x) = 0$. Suppose that none of them is zero.*

Then $\dfrac{1}{\alpha_1}, \dfrac{1}{\alpha_2}, \ldots, \dfrac{1}{\alpha_n}$ are the roots of the equation $x^n f\left(\dfrac{1}{x}\right) = 0$.

Proof : Since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $f(x)=0$, we have

$f(x) = a_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$, where a_0 is the leading coefficient.

For $x \in \mathbf{C} \setminus \{0\}$, $f\left(\dfrac{1}{x}\right) = a_0\left(\dfrac{1}{x}-\alpha_1\right)\left(\dfrac{1}{x}-\alpha_2\right)\ldots\left(\dfrac{1}{x}-\alpha_n\right)$

$$= \frac{a_0}{x^n}(1-\alpha_1x)(1-\alpha_2x)\ldots(1-\alpha_nx)$$

$$= \frac{(-1)^n a_0 \alpha_1 \alpha_2 \ldots \alpha_n}{x^n}\left(x-\frac{1}{\alpha_1}\right)\left(x-\frac{1}{\alpha_2}\right)\ldots\left(x-\frac{1}{\alpha_n}\right)$$

$(\because \alpha_k \neq 0,$ for each $k)$.

Hence $x^n f\left(\frac{1}{x}\right)$ is a polynomial in x and $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$ are the roots of the equation $x^n f\left(\frac{1}{x}\right) = 0$.

For any polynomial $f(x)$, we say that a complex number β is a root of $f(\sqrt{x}) = 0$, if there is a complex number γ such that $\gamma^2 = \beta$ and $f(\gamma) = 0$.

4.4.10 Theorem : *If α is a root of $f(x) = 0$, then α^2 is a root of $f(\sqrt{x}) = 0$.*

Proof : Let α be a root of $f(x) = 0$, Then $f(\alpha) = 0$.

Hence α^2 is a root of $f(\sqrt{x}) = 0$.

4.4.11 Note : For a polynomial $f(x)$, the equation $f(\sqrt{x}) = 0$ need not be a polynomial equation. If it is a polynomial equation, its degree is one half of that of the equation $f(x) = 0$.

Notes 4.4.12, 4.4.13 and 4.4.14 given below are useful in solving problems.

4.4.12 Note : Let ϕ be a one-one map of **C** onto **C**. Let ϕ^{-1} denote the inverse of ϕ. Then α is a root of the equation $f(x) = 0$ iff $\phi(\alpha)$ is a root of the equation $f(\phi^{-1}(y)) = 0$. If $\phi^{-1}(y)$ is a polynomial in y of degree one, then α is a root of $f(x) = 0$ of multiplicity m, iff $\phi(\alpha)$ is a root of $f(\phi^{-1}(y)) = 0$ of multiplicity m.

Theorems 4.4.1, 4.4.3, 4.4.6 and Note 4.4.7 are particular cases of this general transformation.

A polymonial equation whose roots are the translates of the roots $f(x) = 0$ by k is given by $f(x-k) = 0$.

For example, to obtain an algebraic equation whose roots are the translates of the roots of $f(x) = 0$ by 2, we can take $y = \phi(x) = x + 2$, so that $x = \phi^{-1}(y) = y - 2$ and therefore the required transformed equation is $f(y-2) = 0$.

In general when we have to find an algebraic equation with specified roots we usually confine to the monic polynomial equation. As such by "the polynomial equation" we mean the monic polynomial equation.

4.4.13 Note : Let $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ with $a_0 \neq 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $f(x) = 0$. Let $g(x)$ be the polynomial obtained by replacing x^2 by x in $(-1)^n f(x) f(-x)$. Then $\alpha_1^2, \alpha_2^2, \ldots, \alpha_n^2$ are the roots of equation $g(x) = 0$. Further we observe that $(-1)^n f(x) f(-x) = [f_1(x)]^2 - [f_2(x)]^2$,

where $f_1(x) = \sum_{k=0}^{[n/2]} a_{2k} x^{n-2k}$ and $f_2(x) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} a_{2k+1} x^{n-2k-1}$.

4.4.14 Note : Let $f(x)$ be a polynomial of degree $n > 0$ with rational coefficients. Let a and b be positive rational numbers such that $\sqrt{a}, \sqrt{b}$ and $\sqrt{ab}$ are irrational numbers. Suppose that one of the four numbers $\sqrt{a}+\sqrt{b}$, $\sqrt{a}-\sqrt{b}$, $-\sqrt{a}+\sqrt{b}$, $-\sqrt{a}-\sqrt{b}$ is a root of the equation $f(x)=0$. Then all the four numbers are roots of $f(x)=0$.

4.4.15 Remark : The result in Note 4.4.14 fails, if $\sqrt{ab}$ is not an irrational number, in view of the following examples.

4.4.16 Examples

1. $x^2 - 18 = 0$ is a polynomial equation with rational coefficients. Its roots are $3\sqrt{2}$ and $-3\sqrt{2}$. i.e., $\sqrt{2}+\sqrt{8}$ and $-\sqrt{2}-\sqrt{8}$. (Here $a = 2,\ b = 8;\ \sqrt{ab}$ is rational number).

2. $x^2 - 2 = 0$ is a polynomial equation with rational coefficients. Its roots are $\sqrt{2}$ and $-\sqrt{2}$. i.e., $\sqrt{8}-\sqrt{2}$ and $\sqrt{2}-\sqrt{8}$ (Here $a = 8,\ b = 2;\ \sqrt{ab}$ is rational).

4.4.17 Solved Problems

1. Problem : *Find the polynomial equation of degree 4 whose roots are the negatives of the roots of* $x^4 - 6x^3 + 7x^2 - 2x + 1 = 0$.

Solution : Let $f(x) = x^4 - 6x^3 + 7x^2 - 2x + 1$.

By Theorem 4.4.1, the equation $f(-x) = 0$ has the desired property.

We have $f(-x) = (-x)^4 - 6(-x)^3 + 7(-x)^2 - 2(-x) + 1$

$$= x^4 + 6x^3 + 7x^2 + 2x + 1$$

Hence $x^4 + 6x^3 + 7x^2 + 2x + 1 = 0$ is the desired equation.

2. Problem : *Find an algebraic equation of degree 4 whose roots are 3 times the roots of the equation* $6x^4 - 7x^3 + 8x^2 - 7x + 2 = 0$.

Solution : Let $f(x) = 6x^4 - 7x^3 + 8x^2 - 7x + 2$.

By Theorem 4.4.3, the equation $f\left(\frac{x}{3}\right) = 0$ has the desired properties.

We have $f\left(\frac{x}{3}\right) = 6\left(\frac{x}{3}\right)^4 - 7\left(\frac{x}{3}\right)^3 + 8\left(\frac{x}{3}\right)^2 - 7\left(\frac{x}{3}\right) + 2$

$$= \frac{1}{3^4}\left[6x^4 - 21x^3 + 72x^2 - 189x + 162\right]$$

Hence $6x^4 - 21x^3 + 72x^2 - 189x + 162 = 0$ is the desired equation.

3. Problem : *Form the equation whose roots are m times the roots of the equation*

$$x^3 + \frac{1}{4}x^2 - \frac{1}{16}x + \frac{1}{72} = 0 \quad \textit{and deduce the case when } m = 12.$$

Solution : From the Note 4.4.2 and Note 4.4.4, it follows that

$$x^3 + \frac{m}{4}x^2 - \frac{m^2}{16}x + \frac{m^3}{72} = 0$$

$$i.e.,\ x^3 + \frac{m}{2^2}x^2 - \frac{m^2}{2^4}x + \frac{m^3}{2^3.3^2} = 0 \qquad ...(1)$$

is a polynomial equation of degree 3, whose roots are m times those of the given equation. On taking m=12, equation (1) reduces to

$$x^3 + \frac{12}{2^2}x^2 - \frac{12^2}{2^4}x + \frac{12^3}{2^3.3^2} = 0$$

i.e., $x^3 + 3x^2 - 9x + 24 = 0.$

4. Problem : *Find the polynomial equation of degree 5 whose roots are the translates of the roots of* $x^5 + 4x^3 - x^2 + 11 = 0$ by -3.

Solution : By Theorem 4.4.6, the equation

$(x+3)^5 + 4(x+3)^3 - (x+3)^2 + 11 = 0$ has the desired properties.

On simplifying the above equation, we get

$$x^5 + 15x^4 + 94x^3 + 305x^2 + 507x + 353 = 0.$$

The transformed equation can also be obtained by synthetic division

Let $f(x) = x^5 + 4x^3 - x^2 + 11.$

Suppose that $f(x+3) = A_0x^5 + A_1x^4 + A_2x^3 + A_3x^2 + A_4x + A_5$.

Then by Note 4.4.8, the coefficients $A_0, A_1, ..., A_5$ can be obtained as follows

3	1	0	4	–1	0	11
	0	3	9	39	114	342
	1	3	13	38	114	$353 = A_5$
	0	3	18	93	393	
	1	6	31	131	$507 = A_4$	
	0	3	27	174		
	1	9	58	$305 = A_3$		
	0	3	36			
	1	12	$94 = A_2$			
	0	3				
	1	$15 = A_1$				
	0					
	$1 = A_0$					

Therefore the roots of the equation $x^5 + 15x^4 + 94x^3 + 305x^2 + 507x + 353 = 0$ are the translates of the roots of the given equation by –3.

5. Problem : *Find the algebraic equation of degree 4 whose roots are the translates of the roots of* $4x^4 + 32x^3 + 83x^2 + 76x + 21 = 0$ by 2

Solution : By Theorem 4.4.6, the equation

$4(x-2)^4 + 32(x-2)^3 + 83(x-2)^2 + 76(x-2) + 21 = 0$ has the desired properties.

On simlifying the above equation, we get $4x^4 - 13x^2 + 9 = 0$.

Other method : The equation $A_0x^4 + A_1x^3 + \cdots + A_4 = 0$ whose coefficients are obtained by synthetic division as given below, has the desired properties.

$$
\begin{array}{r|rrrrl}
-2 & 4 & 32 & 83 & 76 & 21 \\
 & 0 & -8 & -48 & -70 & -12 \\
\hline
 & 4 & 24 & 35 & 6 & 9 = A_4 \\
 & 0 & -8 & -32 & -6 & \\
\hline
 & 4 & 16 & 3 & 0 = A_3 & \\
 & 0 & -8 & -16 & & \\
\hline
 & 4 & 8 & -13 = A_2 & & \\
 & 0 & -8 & & & \\
\hline
 & 4 & 0 = A_1 & & & \\
 & 0 & & & & \\
\hline
 & 4 = A_0 & & & &
\end{array}
$$

Hence the equation $4x^4 - 13x^2 + 9 = 0$ has the desired properties.

6. Problem : *Find the polynomial equation whose roots are the reciprocals of the roots of the equation* $x^4 + 3x^3 - 6x^2 + 2x - 4 = 0$.

Solution : Let $f(x) = x^4 + 3x^3 - 6x^2 + 2x - 4$.

By Theorem 4.4.9, the equation $x^4 f\left(\frac{1}{x}\right) = 0$ has the desired properties.

Therefore $x^4\left[\frac{1}{x^4} + 3\frac{1}{x^3} - 6\frac{1}{x^2} + \frac{2}{x} - 4\right] = 0$

i.e., $4x^4 - 2x^3 + 6x^2 - 3x - 1 = 0$ is the required equation.

7. Problem : *Find the polynomial equation whose roots are the squares of the roots of* $x^3 - x^2 + 8x - 6 = 0$.

Solution : Let $f(x) = x^3 - x^2 + 8x - 6$.

Then as per the notation introduced in Note 4.4.13, we have

$$f_1(x) = x^3 + 8x \text{ and } f_2(x) = -x^2 - 6$$

Hence $$[f_1(x)]^2 - [f_2(x)]^2 = \left(x^3 + 8x\right)^2 - \left(-x^2 - 6\right)^2$$

$$= x^2\left(x^2 + 8\right)^2 - \left(x^2 + 6\right)^2 \qquad \text{... (1)}$$

Replacing x^2 by x in (1) we get $g(x) = x(x+8)^2 - (x+6)^2$

$$= x^3 + 15x^2 + 52x - 36.$$

The equation $x^3 + 15x^2 + 52x - 36 = 0$ has the desired properties.

4.4.18 To obtain equations with certain missing terms

Given a polynomial $f(x)$ of degree $n > 0$, we now discuss a method of finding $h \in \mathbf{C}$ such that for any given $k \in \{0, 1, ..., n-1\}$ the coefficient of the k^{th} power of x in the polynomial $f(x+h)$ (expressed in powers of x) is zero.

Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$.

Let $h \in \mathbf{C}$. Then $f(x+h) = a_0 (x+h)^n + a_1 (x+h)^{n-1} + \cdots + a_{n-1}(x+h) + a_n$.

Let $k \in \{0, 1, 2, ..., n-1\}$.

On using Binomial Theorem we find that, for $m \in \{k, k+1, ..., n\}$

the coefficient of x^k in $(x+h)^m$ is

$${}^m C_{m-k} h^{m-k} \quad i.e., \quad {}^m C_k \, h^{m-k}.$$

Hence the coefficient of x^k in $f(x+h)$ is

$$\sum_{m=k}^{n} a_{n-m} \, {}^m C_k \, h^{m-k}$$

Therefore, if we choose $h \in \mathbf{C}$ such that

$$\sum_{m=k}^{n} a_{n-m} \, {}^m C_k \, h^{m-k} = 0,$$

then the coefficient of x^k in the expression of $f(x+h)$ in powers of x is zero.

4.4.19 Example : We transform the equation $x^4 + 8x^3 + x - 5 = 0$, so that the term containing the cubic power of x is absent.

Let $f(x) = x^4 + 8x^3 + x - 5$

We have to find h so that the coefficient of x^3 in $f(x+h)$ is zero. We have

$$f(x+h) = (x+h)^4 + 8(x+h)^3 + (x+h) - 5.$$

Then the coefficient of x^3 in $f(x+h)$ is $4h+8$.

Hence we have to choose h such that $4h+8 = 0$ i.e., $h = -2$.

Therefore $f(x-2) = (x-2)^4 + 8(x-2)^3 + x - 7$

$$= x^4 - 24x^2 + 65x - 55 = 0$$ is the required equation.

4.4.20 Definition (Reciprocal equation)

A polynomial $f(x)$ of degree $n > 0$ is said to be reciprocal if $f(0) \neq 0$ and $f(x) = \dfrac{a_0}{f(0)} x^n f\left(\dfrac{1}{x}\right) \forall x \in \mathbf{C} \setminus \{0\}$, *where a_0 is the leading coefficient of $f(x)$.*

If $f(x)$ is a reciprocal polynomial, then the equation $f(x) = 0$ is said to be a reciprocal equation.

4.4.21 Note : If $f(x)$ is a reciprocal polynomial and $\alpha \in \mathbf{C}$, then α is a root of $f(x) = 0$ of multiplicity m, iff $\dfrac{1}{\alpha}$ is a root of $f(x) = 0$ of multiplicity m (follows from Theorem 4.4.9).

4.4.22 Theorem : *Let $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ be a polynomial of degree $n > 0$. Then $f(x)$ is reciprocal* iff $a_{n-k} = a_k$ for $k = 0, 1, 2, \ldots, n$ or $a_{n-k} = -a_k$ for $k = 0, 1, 2, \ldots, n$.

Proof : The coefficient of x^k in $f(x)$ is a_{n-k}.

$$\text{The coefficient of } x^k \text{ in } x^n f\left(\frac{1}{x}\right) = \text{the coefficient of } \left(\frac{1}{x}\right)^{n-k} \text{ in } f\left(\frac{1}{x}\right)$$

$$= \text{the coefficient of } x^{n-k} \text{ in } f(x)$$

$$= a_k$$

$$f(x) \text{ is reciprocal } \Leftrightarrow f(0) \neq 0 \text{ and } f(x) = \frac{a_0}{f(0)} x^n f\left(\frac{1}{x}\right) \forall x \in \mathbf{C} \setminus \{0\}$$

$$\Leftrightarrow a_n \neq 0 \text{ and } a_{n-k} = \frac{a_0}{a_n} a_k \text{ for } k = 0, 1, 2, \ldots, n$$

$$\Leftrightarrow a_{n-k}\, a_n = a_0\, a_k \text{ for } k = 0, 1, 2, \ldots, n \ (\because a_0 \neq 0)$$

$$\Leftrightarrow a_n^2 = a_0^2 \text{ and } a_{n-k}\, a_n = a_0\, a_k \text{ for } k = 1, 2, \ldots, n$$

$$\Leftrightarrow a_n = \pm a_0 \text{ and } a_{n-k}\, a_n = a_0\, a_k \text{ for } k = 1, 2, \ldots, n$$

$$\Leftrightarrow a_{n-k} = a_k \text{ for } k = 0, 1, 2, \ldots, n$$

$$\text{or } a_{n-k} = -a_k \text{ for } k = 0, 1, 2, \ldots, n.$$

4.4.23 Definition

A reciprocal polynomial $f(x)$ of degree n with leading coefficient a_0 is said to be of class one or class two according as $f(0)$ is equal to a_0 or $-a_0$. If $f(x)$ is a reciprocal polynomial, then the equation $f(x)=0$ is said to be a reciprocal equation of class one or class two according as $f(x)$ is a reciprocal polynomial of class one or class two.

4.4.24 Note

1. For an odd degree reciprocal equation of class one, -1 is a root and for an odd degree reciprocal equation of class two, 1 is a root.
2. For an even degree reciprocal equation of class two, 1 and -1 are roots.
3. To solve a reciprocal equation of order $2m$, divide the equation by x^m and put $x+\frac{1}{x}=y$ or $x-\frac{1}{x}=y$ according as the equation is of class one or class two. The degree of the transformed equation is m.
4. Let $f(x)=0$ be an odd degree reciprocal equation. To find the roots of it, divide $f(x)$ by $(x+1)$ or $(x-1)$ according as the equation is of class one or class two. Let $Q(x)$ be the quotient obtained. Then $f(x)=(x+1)\,Q(x)$ or $f(x)=(x-1)\,Q(x)$ according as the equation is of class one or class two and $Q(x)$ is even degree reciprocal polynomial. The roots of $Q(x)=0$ can be obtained by the procedure described in the above Note 4.4.24 (3).

4.4.25 Solved Problems

1. Problem : *Show that* $2x^3+5x^2+5x+2=0$ *is a reciprocal equation of class one.*

Solution : Let $f(x)=2x^3+5x^2+5x+2=0$. Then 2 is the leading coefficient

and $$f\left(\frac{1}{x}\right)=2\left(\frac{1}{x}\right)^3+5\left(\frac{1}{x}\right)^2+5\left(\frac{1}{x}\right)+2.$$

Hence $$\frac{a_0}{f(0)}x^3 f\left(\frac{1}{x}\right)=2+5x+5x^2+2x^3=f(x), \text{ where } a_0=2=f(0)$$

Therefore the given equation is a reciprocal equation of class one.

2. Problem : *Solve the equation* $4x^3-13x^2-13x+4=0$.

Solution : The given equation is an odd degree reciprocal equation of class one.

By Note 4.4.24(1), -1 is a root of this equation.

Therefore $(x+1)$ is a factor of $4x^3-13x^2-13x+4$.

Hence, on dividing this expression by $(x+1)$, we get

$$4x^3 - 13x^2 - 13x + 4 = (x+1)(4x^2 - 17x + 4).$$

Now the roots of the equation $4x^2 - 17x + 4 = 0$ are $\frac{1}{4}$ and 4.

Therefore $-1, \frac{1}{4}, 4$ are the roots of the given equation.

3. Problem : *Solve the equation* $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$.

Solution : We observe that the given equation is an odd degree reciprocal equation of class two. By Note 4.4.24(1), 1 is a root of this equation. Therefore $(x-1)$ is a factor of $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1$.

On dividing this expression by $(x-1)$, we get

$$x^4 - 4x^3 + 5x^2 - 4x + 1 \text{ as the quotient.}$$

Now we have to solve the equation $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$.

On dividing this equation by x^2, we get

$$x^2 - 4x + 5 - \frac{4}{x} + \frac{1}{x^2} = 0$$

i.e., $\left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 5 = 0$.

Put $x + \frac{1}{x} = y$, so that $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = y^2 - 2$.

Then the above equation reduces to $\left(y^2 - 2\right) - 4y + 5 = 0$. *i.e.,* $y^2 - 4y + 3 = 0$.

$$y^2 - 4y + 3 = 0 \Leftrightarrow (y-3)(y-1) = 0 \Leftrightarrow y = 3 \text{ or } 1.$$

Therefore $x + \frac{1}{x} = 3$ or $x + \frac{1}{x} = 1$

i.e., $x^2 - 3x + 1 = 0$ or $x^2 - x + 1 = 0$.

The roots of these equations are $x = \frac{3 \pm \sqrt{5}}{2}$ and $x = \frac{1 \pm i\sqrt{3}}{2}$ respectively.

Hence the roots of the given equation are $1, \frac{3 \pm \sqrt{5}}{2}$ and $\frac{1 \pm i\sqrt{3}}{2}$.

4. Problem : *Solve the equation* $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0.$

Solution : We observe that the given equation is an even degree reciprocal equation of class one. On dividing both sides of the given equation by x^2, we get

$$6x^2 - 35x + 62 - \frac{35}{x} + \frac{6}{x^2} = 0$$

i.e., $$6\left(x^2 + \frac{1}{x^2}\right) - 35\left(x + \frac{1}{x}\right) + 62 = 0.$$

Put $\left(x + \frac{1}{x}\right) = y$, so that $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = y^2 - 2.$

Then the above equation reduces to

$$6(y^2 - 2) - 35y + 62 = 0$$

i.e., $$6y^2 - 35y + 50 = 0$$

i.e., $$(2y - 5)(3y - 10) = 0.$$

Hence the roots of $6y^2 - 35y + 50 = 0$ are $\frac{5}{2}$ and $\frac{10}{3}$.

Therefore $x + \frac{1}{x} = \frac{5}{2}$ and $x + \frac{1}{x} = \frac{10}{3}$

i.e., $2x^2 - 5x + 2 = 0$ and $3x^2 - 10x + 3 = 0.$

The roots of these equations are respectively

$$x = \frac{5 \pm \sqrt{25 - 16}}{2.2} \quad \text{and} \quad x = \frac{10 \pm \sqrt{100 - 36}}{2.3}$$

i.e., $$x = \frac{5 \pm 3}{4} \quad \text{and} \quad x = \frac{10 \pm 8}{6}$$

i.e., $$x = 2, \frac{1}{2} \quad \text{and} \quad x = 3, \frac{1}{3}.$$

Hence the roots of the given equation are $\frac{1}{3}, \frac{1}{2}, 2$ and 3.

5. Problem : *Solve the equation* $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0.$

Solution : We observe that the given equation is an even degree reciprocal equation of class two. By Note 4.4.24(2), $+1$ and -1 are the roots of this equation. Hence $(x + 1)$ and $(x - 1)$ are the factors of the given equation.

Let $f(x) = 6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$.

On dividing this expression by $(x+1)$ and then by $(x-1)$, we get

$$f(x) = (x^2 - 1)(6x^4 - 25x^3 + 37x^2 - 25x + 6).$$

Now we have to solve the equation

$$6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0.$$

On dividing both sides of this equation by x^2, we obtain

$$6x^2 - 25x + 37 - \frac{25}{x} + \frac{6}{x^2} = 0$$

i.e., $6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x + \frac{1}{x}\right) + 37 = 0$.

Put $x + \frac{1}{x} = y$, so that $\left(x^2 + \frac{1}{x^2}\right) = \left(x + \frac{1}{x}\right)^2 - 2 = y^2 - 2.$

Then the above equation reduces to $6(y^2 - 2) - 25y + 37 = 0$

i.e., $6y^2 - 25y + 25 = 0$

i.e., $6y^2 - 15y - 10y + 25 = 0$

i.e., $3y(2y - 5) - 5(2y - 5) = 0$

i.e., $(2y - 5)(3y - 5) = 0$

i.e., $2\left(x + \frac{1}{x}\right) - 5 = 0$ or $3\left(x + \frac{1}{x}\right) - 5 = 0$

i.e., $2x^2 - 5x + 2 = 0$ or $3x^2 - 5x + 3 = 0$

i.e., $(x - 2)(2x - 1) = 0$ or $3x^2 - 5x + 3 = 0$

i.e., $x = \frac{1}{2}, 2$ or $x = \frac{5 \pm \sqrt{25 - 36}}{6}$

Hence the roots of the given equation are $\pm 1, \frac{1}{2}, 2, \frac{5 \pm i\sqrt{11}}{6}$.

Exercise 4(d)

I. 1. Find the algebraic equation whose roots are 3 times the roots of $x^3+2x^2-4x+1=0$.

2. Find the algebraic equation whose roots are 2 times the roots of
$x^5-2x^4+3x^3-2x^2+4x+3=0$.

3. Find the transformed equation whose roots are the negatives of the roots of
$x^4+5x^3+11x+3=0$.

4. Find the transformed equation whose roots are the negatives of the roots of
$x^7+3x^5+x^3-x^2+7x+2=0$.

5. Find the polynomial equation whose roots are the reciprocals of the roots of
$x^4-3x^3+7x^2+5x-2=0$.

6. Find the polynomial equation whose roots are the reciprocals of the roots of
$x^5+11x^4+x^3+4x^2-13x+6=0$.

II. 1. Find the polynomial equation whose roots are the squares of the roots of
$x^4+x^3+2x^2+x+1=0$.

2. Form the polynomial equation whose roots are the squares of the roots of
$x^3+3x^2-7x+6=0$.

3. Form the polynomial equation whose roots are the cubes of the roots of
$x^3+3x^2+2=0$.

III. 1. Find the polynomial equation whose roots are the translates of those of the equation
$x^4-5x^3+7x^2-17x+11=0$ by -2.

2. Find the polynomial equation whose roots are the translates of those of the equation
$x^5-4x^4+3x^2-4x+6=0$ by -3.

3. Find the polynomial equation whose roots are the translates of the roots of the equation
$x^4-x^3-10x^2+4x+24=0$ by 2.

4. Find the polynomial equation whose roots are the translates of the roots of the equation
$3x^5-5x^3+7=0$ by 4.

5. Transform each of the following equations into ones in which the coefficients of the second highest power of x is zero and also find their transformed equations.

(i) $x^3 - 6x^2 + 10x - 3 = 0$ (ii) $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$

(iii) $x^3 - 6x^2 + 4x - 7 = 0$ (iv) $x^3 + 6x^2 + 4x + 4 = 0$

6. Transform each of the following equations into ones in which the coefficients of the third highest power of x is zero.

(i) $x^4 + 2x^3 - 12x^2 + 2x - 1 = 0$ (ii) $x^3 + 2x^2 + x + 1 = 0$

7. Solve the following equations.

(i) $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$ (ii) $2x^5 + x^4 - 12x^3 - 12x^2 + x + 2 = 0.$

Key Concepts

❖ If n is a non-negative integer and $a_0, a_1, a_2, ..., a_n$ are real or complex numbers and $a_0 \neq 0$, then the expression $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ is called a polynomial in x of degree n.

❖ If $f(x)$ is a polynomial of degree $n > 0$, then the equation $f(x) = 0$ is called an algebraic equation or polynomial equation of degree n.

❖ A complex number α is said to be zero of a polynomial $f(x)$ or a root of the equation $f(x) = 0$, if $f(\alpha) = 0$.

❖ Every non-constant polynomial equation has atleast one root.

❖ Relation between the roots and the coefficients of an equation :

(i) If $\alpha_1, \alpha_2, \alpha_3$ are the roots of $x^3 + p_1x^2 + p_2x + p_3 = 0$

$s_1 = \alpha_1 + \alpha_2 + \alpha_3 = -p_1$

$s_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = p_2$

$s_3 = \alpha_1\alpha_2\alpha_3 = -p_3$

(ii) If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of $x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$, then

$s_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -p_1$

$s_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_4 = p_2$

$$s_3 = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \alpha_3\alpha_4\alpha_1 + \alpha_1\alpha_2\alpha_4 = -p_3$$

$$s_4 = \alpha_1\alpha_2\alpha_3\alpha_4 = p_4$$

❖ For a cubic equation, when the roots are

(i) in A.P., then they are taken as $a-d,\ a,\ a+d$

(ii) in G.P., then they are taken as $\dfrac{a}{d}, a,\ ad$

(ii) in H.P., then they are taken as $\dfrac{1}{a-d}, \dfrac{1}{a}, \dfrac{1}{a+d}$

❖ For a biquadratic equation, if the roots are

(i) in A.P., then they are taken as $a-3d,\ a-d, a+d,\ a+3d$

(ii) in G.P., then they are taken as $\dfrac{a}{d^3}, \dfrac{a}{d}, ad, ad^3$

(ii) in H.P., then they are taken as $\dfrac{1}{a-3d}, \dfrac{1}{a-d}, \dfrac{1}{a+d}, \dfrac{1}{a+3d}$

❖ To find a root of $f(x)=0$, we have to find out a value of x, for which $f(x)=0$. Some times we can do this by inspection. This method is known as trial and error method.

❖ (i) For a polynomial equation with rational coefficients, irrational roots occur in pairs.

(ii) For a polynomial equation with real coefficients, complex roots occur in pairs.

❖ If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the equation $f(x)=0$, then $\alpha_1-h, \alpha_2-h, \ldots, \alpha_n-h$ are the roots of the equation $f(x+h)=0$ and $\alpha_1+h,\ \alpha_2+h, \ldots, \alpha_n+h$ are the roots of the equation $f(x-h)=0$.

❖ If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$, then the transformed equation whose roots are the reciprocals of the roots of $f(x)=0$ is

$$\phi(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0.$$

❖ If an equation is unaltered by changing x into $\dfrac{1}{x}$, then it is a reciprocal equation.

Historical Note

We have evidence to show that people around 1700 B.C. were looking beyond arithmetic into the vistas of algebra and succeeded in constructing equations. Solving equations in algebra is greatly simplified by the use of formulas of symbols. It was the 17th Century *Rene Descartes* who systemised to a large extent the symbolism in the theory of equations.

Before an algebraic notation was developed and before the birth of the idea that equations could be classified and that each class of equations had a general solution, algebraic problems were like riddles, each equation was treated as a separate case with its special solution. The first scholars who fully explained their methods for solving algebraic problems - linear, quadratic and cubic - were the Greeks. *Diophantus* is remembered as the *father of algebra.* Later we find significant contributions to the theory of equations by scholars of Arabian and Indian origin, followed by mathematicians from Europe, in particular from Italy.

Pavuluri Mallana's beautiful verse narrated below unfolds a quadratic equation to be solved:

" The roar of the clouds thrill a group of peacocks on a hill, two sevenths of them in ecstasy danced, twice the square root of them ran in gay abandon, the remaining twenty one were just witnessing the whole scenario. How many peacocks were there in the group ?".

Answers

Exercise 4(a)

I. 1. (i) $x^3-3x^2-x+3=0$ (ii) $x^4-8x^3+25x^2-46x+40=0$

(iii) $x^4-6x^3+14x^2-22x+5=0$ (iv) $x^6-8x^4+16x^2=0$

(v) $x^4-9x^3+22x^2-6x-20=0$ (vi) $4x^4+12x^3-x^2-15x=0$

2. $\frac{7}{4}$ 3. 4

4. $-\frac{3}{2}$ 5. -5 6. 36

7. (i) 16, 86, 176, 105 (ii) $\frac{1}{4}, -\frac{27}{8}, -\frac{3}{4}, \frac{9}{8}$

II. 1. $3, \ -2$

2. (i) -7 (ii) -6

3. (i) $(p^2-2q)/r^2$ (ii) $\dfrac{pq}{r}-3$

(iii) $3p^3-16pq+64r$ (iv) $q^3-3pqr+3r^2$

III. 1. $x^3-28x^2+245x-650=0$ 2. $x^3-42x^2+441x-400=0$

Exercise 4(b)

I. 1. $4, \ -4, \ 3$ 2. $pq=r$

4. $q^3=p^3r$

II. 1. $\dfrac{1}{3},\dfrac{1}{3},1$ 2. $\dfrac{1}{2},1,-3$

3. $6,4,-1$

4. (i) $-\dfrac{3}{2},\dfrac{3}{2},\dfrac{9}{2}$ (ii) $4,1,-2$

5. (i) $6, \ 2, \ \dfrac{2}{3}$ (ii) $\dfrac{8}{9}, \ -\dfrac{2}{3}, \ \dfrac{1}{2}$

6. (i) $1, \ \dfrac{1}{2}, \ \dfrac{1}{3}$ (ii) $1, \ \dfrac{1}{3}, \ \dfrac{1}{5}$

7. (i) $3, \ 3, \ \pm 2i$ (ii) $-2, \ -2, -2, \ \dfrac{2}{3}$

III. 1. $3, \ 2, -2, -4$ 2. $\sqrt{3}, \ -\sqrt{3}, \ \dfrac{3}{4}, -\dfrac{1}{2}$ 3. $-\dfrac{4}{3}, \ -\dfrac{3}{2}, -\dfrac{5}{3}$

4. $2b^3+a^2d=3abc, ad^2-eb^2=0$ 5. (i) 1 (ii) 1, 2 6. $\dfrac{3}{2}, \ \dfrac{3}{2}, -\dfrac{1}{2}$

Exercise 4(c)

I. 1. (i) $x^4-6x^3+23x^2-34x+26=0$ (ii) $x^4-7x^3+18x^2-22x+12=0$

(iii) $x^4+4=0$ (iv) $x^4-4x^3+8x^2-8x+4=0$

2. (i) $x^4-10x^3-19x^2+480x-1392=0$ (ii) $x^4-12x^3+72x^2-312x+676=0$

(iii) $x^4-8x^2+36=0$ (iv) $x^4-2x^2+25=0$

II. 1. $-2 \pm \sqrt{3}$; $1 \pm \sqrt{-1}$ 2. $2 \pm \sqrt{-7}$; $-\frac{8}{3}$ 3. $2 \pm i\sqrt{3}$; $-2 \pm i$

4. $2 \pm \sqrt{3}$; $1 \pm i$ 5. $-2 \pm i\sqrt{7}$; $2 \pm i\sqrt{3}$ 6. $-1, -7$; $2 \pm \sqrt{3}$

7. $\pm\sqrt{2} \pm \sqrt{5}$; $\frac{4}{3}$ 8. $2, 3, 2 \pm \sqrt{3}$

Exercise 4(d)

I. 1. $x^3 + 6x^2 - 36x + 27 = 0$ 2. $x^5 - 4x^4 + 12x^3 - 16x^2 + 64x + 96 = 0$

3. $x^4 - 5x^3 - 11x + 3 = 0$ 4. $x^7 + 3x^5 + x^3 + x^2 + 7x - 2 = 0$

5. $2x^4 - 5x^3 - 7x^2 + 3x - 1 = 0$ 6. $6x^5 - 13x^4 + 4x^3 + x^2 + 11x + 1 = 0$

II. 1. $x^4 + 3x^3 + 4x^2 + 3x + 1 = 0$ 2. $x^3 - 23x^2 + 13x - 36 = 0$

3. $x^3 + 33x^2 + 12x + 8 = 0$

III. 1. $x^4 + 3x^3 + x^2 - 17x - 19 = 0$ 2. $x^5 + 11x^4 + 42x^3 + 57x^2 - 13x - 60 = 0$

3. $x^4 - 9x^3 + 40x^2 - 80x + 80 = 0$

4. $3x^5 - 60x^4 + 475x^3 - 1860x^2 + 3600x - 2745 = 0$

5. (i) $x^3 - 2x + 1 = 0$ (ii) $x^4 - 4x^2 + 1 = 0$ (iii) $x^3 - 8x - 15 = 0$ (iv) $x^3 - 8x + 12 = 0$

6. (i) $x^4 + 6x^3 - 12x - 8 = 0$ or $x^4 - 6x^3 + 42x - 53 = 0$

(ii) $x^3 - x^2 + 1 = 0$ or $27x^3 + 27x^2 + 23 = 0$

7. (i) $3 \pm 2\sqrt{2}, 2 \pm \sqrt{3}$ (ii) $-\frac{1}{2}, -1, -2, \frac{3 \pm \sqrt{5}}{2}$

Chapter 5

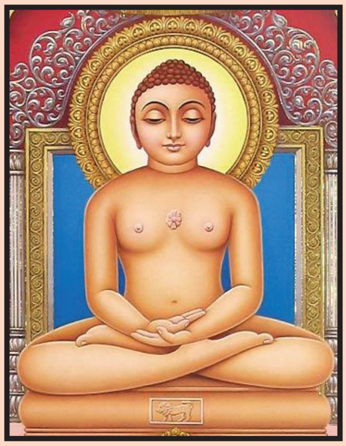
Permutations and Combinations

" Mathematical proofs, like Diamonds, are hard and clear and will be touched with nothing but strict reasoning" **- *John Locke***

Introduction

The first known use of permutations and combinations goes back to 6^{th} century B.C. when '*Susruta*' in his medicinal work '*Susruta Samhita*' finds 63 combinations out of 6 different tastes by taking one at a time, two at a time etc. Later in the 3^{rd} century B.C., a Sanskrit scholar '*Pingala*' in his book '*Chandassastra*' used permutations and combinations to determine the number of combinations of a given number of letters by taking one at a time, two at a time etc. The concept of permutations and combinations was treated as a self contained topic in mathematics under the name "*Vikalpa*" by renowned mathematician '*Mahavira*' in 9^{th} century A.D. The credit of stating several important theorems and results on the subject matter of permutations and combinations goes to the renowned scholar "*Bhaskaracharya*". He treated this topic under the name '*Anka Vyastha*' in his famous book '*Leelavathi Ganitham*".

The theory of permutations and combinations in the present sense first appeared in the book 'Ars Conjectandi' written by the renowned mathematician "Jakob Bernoulli" in 17th Century A.D. which was published in 1713 A.D. after his death.



Mahaviracharya
(9th century)

Mahavira *was a 9th century Indian mathematician from Karnataka. He was the author of Ganita Saara Sangraha. He liberated Mathematics from Astronomy.*

We must have come across situations like choosing five questions out of eight questions in a question paper or which items to be chosen from the menu card in a hotel etc. We discuss such situations in this chapter. This chapter 'permutations and combinations' is an important chapter in algebra in view of a number of applications in day - to - day life and in the theory of probability. While learning **'permutations and combinations'**, we should be in a position to clearly see whether the concept of a permutation or the concept of a combination is applicable in a given situation. In general, a combination is only a selection while a permutation involves two steps, namely, selection and arrangement. For example, forming a three digit number using the digits 1, 2, 3, 4, 5 is a **'permutation'**. This involves two steps. In the first step we select three digits, say 2, 4, 5. In the second step, we arrange them to form a three digit number such as 245, 452, 542 etc. Forming a set with three elements using the digits 1, 2, 3, 4, 5 is a **'combination'**. This involves only one process, namely, selection of three elements, say 2, 4, 5. Then the element set formed is $\{2, 4, 5\}$ which is same as the sets $\{4, 5, 2\}$ $\{5, 4, 2\}$ etc. Thus, whenever there is importance to the arrangement or order in which the objects are placed, then it is a **'permutation'** and if there is no importance to the arrangement or order, but only selection is required, then it is a **'combination'**. These notions will help us to arrive at the number of arrangements or combinations without actually counting them.

Before going into formal definitions, we introduce **factorial** notation, which is required to calculate the number of permutations or combinations. If n is a positive integer, we define $n!$ (read as n factorial) by mathematical induction as follows.

$$1! = 1$$

and $$n! = n\,.\,((n-1)!) \text{ if } n > 1.$$

For example,

$$2! = 2\,(1!) = 2$$

$$3! = 3\,(2!) = 3.2 = 6$$

$$4! = 4\,(3!) = 4.6 = 24$$

$$5! = 5\,(4!) = 5.24 = 120 \text{ etc.}$$

By convention, we define $0! = 1$

Throughout this chapter the letters n, r denote nonnegative integers unless otherwise mentioned.

5.1 Fundamental Principle of Counting - Linear and Circular permutations

Before giving formal definitions of linear and circular permutations, we first learn about the **"Fundamental Principle of Counting"**, which plays a very crucial role in the development of the theory of permutations and combinations.

5.1.1 Fundamental principle of Counting

If a work w_1 can be performed in 'm' different ways and a second work w_2 can be performed (after w_1 has been performed in any one of the 'm' ways) in n different ways, then the two works (one after the other) can be performed in 'mn' different ways.

This principle can be easily understood with the help of the following two examples.

5.1.2 Example : If a man has 4 different coloured trousers T_1, T_2, T_3, T_4 and three different coloured shirts S_1, S_2, S_3, then he can select a pair (a trouser and a shirt) in $4\times3 = 12$ different ways as explained below.

In this example, we take w_1 as selecting a trouser which can be performed in 4 ways and w_2 as selecting a shirt which can be performed in 3 ways. Hence, by the fundamental principle, he can select a pair in $4\times3 = 12$ different ways. They are

$T_1 S_1$	$T_1 S_2$	$T_1 S_3$
$T_2 S_1$	$T_2 S_2$	$T_2 S_3$
$T_3 S_1$	$T_3 S_2$	$T_3 S_3$
$T_4 S_1$	$T_4 S_2$	$T_4 S_3$

5.1.3 Example : If there are four different modes of transport available to travel from Hyderabad (HYD) to Chennai (CH), namely, bus, car, train and aeroplane (we denote these by M_1, M_2, M_3, M_4 respectively) and three different modes of transport from Chennai to Bangalore (BG), namely, bus, train, aeroplane (we denote these by N_1, N_2, N_3 respectively), then how many different modes of transport are available for a person to travel from Hyderabad to Bangalore (via Chennai)?

Solution : Here the work w_1 is to travel from HYD to CH, which can be performed in 4 different ways and the work w_2 is to travel from CH to BG, which can be performed in 3 different ways. Therefore, by the fundamental principle, the two works can be done in $4\times3 = 12$ ways. That is, a person can travel from HYD to BG (via CH) in 12 different ways. They are

$M_1 N_1$	$M_1 N_2$	$M_1 N_3$
$M_2 N_1$	$M_2 N_2$	$M_2 N_3$
$M_3 N_1$	$M_3 N_2$	$M_3 N_3$
$M_4 N_1$	$M_4 N_2$	$M_4 N_3$

In this,

M_1N_1 means that the person travels by bus from HYD to CH and again by bus from CH to BG

M_2N_3 means that the person travels by car from HYD to CH and by aeroplane from CH to BG

M_3N_2 means travelling from HYD to CH by train and CH to BG by train again etc.

Now we give the definition of a **linear permutation**.

5.1.4 Definition

From a given finite set of elements (similar or not) selecting some or all of them and arranging them in a line is called a ***'linear permutation'*** *or simply a* ***'permutation'***.

This definition is explained in the following illustrations.

5.1.5 Examples

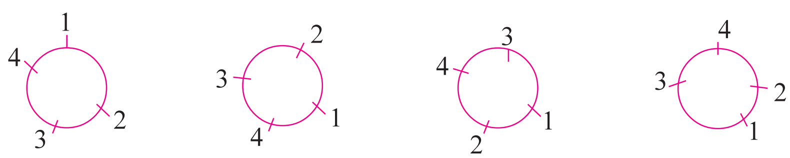
1. **Example :** From the letters of the word MINT, **two** letter permutations are MI, IM, MN, NM, MT, TM, IN, NI, IT, TI, NT, TN
2. **Example :** From the letters of the word RUNNING
   (i) permutations with **three** letters are RUN, UNN, GUR,GNU, NNN etc.
   (ii) permutations with **Four** letters are RUNN, UNIG, NNIN, GNUN, etc.
3. **Example :** Using the digits 1, 2, 3, 4, 5
   (i) permutations with **two** digits (or two digited numbers) are 12, 13, 32, 52, 53, 45 etc.
   (ii) permutations with **three** digits are 123, 324, 513, 352 etc.
   (iii) permutations with **four** digits are 1234, 4351, 5124 etc.

Next, we define circular permutation in the following.

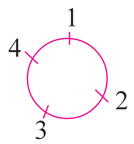
5.1.6 Definition

From a given finite set of things (similar or not) choosing some or all of them and arranging them around a circle is called a ***'circular permutation'***

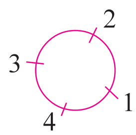
5.1.7 Example : Some of the circular permutations formed using the digits 1, 2, 3, 4 are



The important difference between a circular permutation and a linear permutation is that a linear permutation has a first place (also a last place), where as a circular permutation has no starting place. It can be treated as starting from any one of the elements in it. But how the other elements are arranged relative to this (starting) element is to be taken into consideration. Thus the linear permutations 1234, 2341, 3412, 4123 give rise to only one circular permutation. That is,



Similarly, the linear permutations 1432, 4321, 3214, 2143 give rise to only one circular permutation given below.



Thus in a circular permutation where the first element is placed is not important but how the remaining elements are arranged relative to that element is important.

In some cases like garlands of flowers, chains of beads etc., there is no distinction between the clock-wise and anti clock - wise arrangements of the same circular permutation. They will be treated as a single circular permutation. In such cases, the two circular permutations described above will be treated as a single circular permutation.

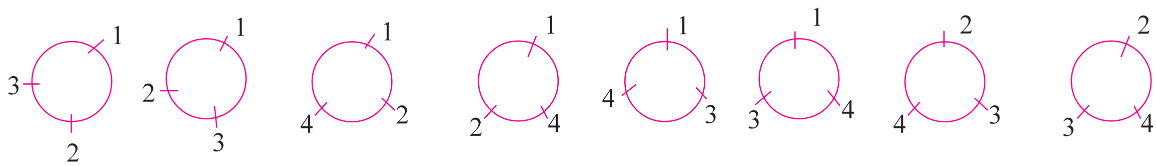
5.1.8 Example : Write all possible (i) linear (ii) circular permutations using the digits 1, 2, 3, 4 taken **three** at a time.

Solution : (i) Required linear permutations are

123	124	134	234
231	241	341	342
312	412	413	423
132	142	143	243
321	421	431	432
213	214	314	324

Thus the number of linear permutations that can be formed using the digits 1, 2, 3, 4 taken **three** at a time is 24.

(ii) Required circular permutations are



Hence, the number of circular permutations that can be formed using the digits 1, 2, 3, 4 taken **three** at a time is 8.

5.2 Permutations of n dissimilar things taken r at a time

Hereafter by a permutation we mean a linear permutation in which no object is used more than once (that is, a permutation without repetition). If repetition is allowed any where, it will be clearly mentioned.

In example 5.1.8 of the previous section, we have exhibited that the number of permutations of 4 dissimilar things taken 3 at a time 24. But if the number of given things and (or) the number of things to be arranged is large, then it is not easy to enumerate the permutations like in example 5.1.8. Hence we develope a formula to find the number of such permutations in the following.

5.2.1 Theorem : *If n, r are positive integers and $r \le n$, then the number of permutations of n dissimilar things taken r at a time is*

$$n(n-1)(n-2)\ldots(n-r+1). \qquad \textit{That is, } \prod_{k=0}^{r-1}(n-k)$$

Proof : We prove this theorem by using induction on n. If $n = 1$, then $r = 1$. In this case, the theorem is trivial. Assume $n \ge 2$ and suppose that the theorem is true for $n-1$. That is, for any s, $1 \le s \le (n-1)$, the number of permutations of $(n-1)$ dissimilar things taken 's' at a time is

$$(n-1)(n-2)\ldots((n-1)-(s-1)). \text{ That is, } \prod_{k=0}^{s-1}((n-1)-k).$$

Now suppose n dissimilar things (objects) are given. Observe that the number of the required permutations is equal to the number of ways of filling 'r' blanks using the given n things with one object in each blank. Let us take r blanks.

$$\underset{1}{\square} \quad \underset{2}{\square} \quad \underset{3}{\square} \; \text{-----} \; \underset{r-1}{\square} \quad \underset{r}{\square}$$

To fill the first blank, we can use any one of the given n things. Thus, the first blank can be filled in 'n' different ways.

After filling up the first blank, we are left with $(n-1)$ things and $(r-1)$ blanks. These $(r-1)$ blanks are to be filled with these $(n-1)$ things. By induction hypothesis (taking $s = r-1$), the number of such permutations is equal to $\prod_{k=0}^{r-2}((n-1)-k)$.

Alternate Proof : To prove that the number of permunations of n dissimilar things taken 'r' at at time

$= n(n-1)(n-2)\ldots(n-r+1)$ ..... (1)

(i) For r = 1, number of permutations $= n = n - 1 + 1$

(ii) Assume that number of permutations $= n(n-1)(n-2)\ldots(n-k+1)$ ..... (2) for $n = k$.

(iii) For $n = k+1$, number of permutations = number of ways of filling k places $\times$ number of ways of filling $(k+1)^{\text{th}}$ place $= n(n-1)(n-2) \ldots (n-k+1) \times (n-k)$ ($\because$ remaining things are $(n-k)$

$= n(n-1)(n-2) \ldots (n-(k+1)+1)$

$= n(n-1) \ldots (n-r+1)$. Hence by mathematical induction, the theorem follows.

Here we have performed two works w_1 and w_2. The work w_1 is filling up the first blank and the work w_2 is filling up the remaining $(r-1)$ blanks using the remaining $(n-1)$ things. Hence, by the fundamental principle (5.1.1) the number of ways in which the two works w_1 and w_2 can be performed, (that is, the r blanks can be filled using the given n things) is

$$n \cdot \prod_{k=0}^{r-2} ((n-1)-k)$$

$$= n(n-1)(n-2) \ldots ((n-1)-(r-2))$$

$$= n(n-1)(n-2) \ldots (n-r+1)$$

$$= \prod_{k=0}^{r-1} (n-k).$$

Hence, by mathematical induction, the theorem follows.

Notation : The number of permutations of n dissimilar things taken r at a time is denoted by ${}^nP_r$ or $P(n, r)$. However, we use the notation ${}^nP_r$ only. Thus, for $1 \le r \le n$,

$\boxed{{}^nP_r = n(n-1) \ldots (n-r+1)}$ and we write ${}^nP_0 = 1$ by convention.

5.2.2 Formula : If $n \ge 1$ and $0 \le r \le n$, then $\boxed{{}^nP_r = \dfrac{n!}{(n-r)!}}$

For $1 \le r \le n$, from Theorem 5.2.1, we get

$$\begin{aligned} {}^nP_r &= n(n-1) \ldots (n-r+1) \\ &= \frac{[n(n-1)\ldots(n-r+1)][(n-r)(n-r-1)\ldots2.1]}{(n-r).(n-r-1)\ldots2.1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

By convention, ${}^nP_0 = 1 = \dfrac{n!}{(n-0)!}$

5.2.3 Note : $\boxed{{}^nP_n = n! \text{ and } {}^nP_0 = 1}$

5.2.4 Theorem : For $1 \le r \le n$, ${}^nP_r = n \,.\, {}^{(n-1)}P_{(r-1)}$

First method : We know that the number of ways of filling up r blanks with n dissimilar things is $^{n}P_{r}$. The first blank can be filled with any one of the given n things in n ways.

Now we can fill the remaining $(r-1)$ blanks with the remaining $(n-1)$ things in $^{(n-1)}P_{(r-1)}$ ways. Thus the number of ways of filling up r blanks with n things is $n\,.\,^{(n-1)}P_{(r-1)}$

That is, $$^{n}P_{r} = n\,.\,^{(n-1)}P_{(r-1)}$$

Second Method : $n\,.\,^{(n-1)}P_{(r-1)} = n.\dfrac{(n-1)!}{((n-1)-(r-1))!} = \dfrac{n((n-1)!)}{(n-r)!} = \dfrac{n!}{(n-r)!} = {}^{n}P_{r}$.

5.2.5 Note : From the above Formula we also get that

$$\begin{aligned} ^{n}P_{r} &= n\,.\,^{(n-1)}P_{(r-1)} \\ &= n\,.\,(n-1)\,.\,^{(n-2)}P_{(r-2)} \\ &= n\,.\,(n-1)\,.\,(n-2)\,.\,^{(n-3)}P_{(r-3)} \text{ etc.} \end{aligned}$$

5.2.6 Example : Find the number of permutations of 4 dissimilar things taken 3 at a time.

Solution : From Formula 5.2.2, the required number of permutations is

$$^{4}P_{3} = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24.$$

5.2.7 Note

In example 5.1.8, we exhibited these permutations and then counted them as 24. Here we have used the formula.

5.2.8 Example : Find the number of all 4 letter words that can be formed using the letters of the word EQUATION. How many of these words begin with E? How many end with N? How many begin with E and end with N?

Solution : The word EQUATION has 8 distinct letters. We have to fill up 4 places using these 8 letters.

This can be done in $^{8}P_{4} = 8.7.6.5 = 1680$ ways. Hence the number of 4 letter words that can be formed using the letters of the word EQUATION is 1680.

Words beginning with E : Fill the first place with E as shown below.

E			

Now we are left with 7 letters and 3 places. They can be filled in

$$^{7}P_{3} = 7.6.5 = 210 \text{ ways.}$$

Thus the number of 4 letter words that begin with E is 210.

Words ending with N : This can be done in the same way as above. First fill the last place with N as shown below

$$\square \quad \square \quad \square \quad \boxed{N}$$

Then the remaining 3 places with the remaining 7 letters can be filled in ${}^{7}P_{3} = 210$ ways. Hence the number of 4 letter words ending with N is 210.

Words beginning with E and ending with N : Fill the first place with E and the last with N as shown below

$$\boxed{E} \quad \square \quad \square \quad \boxed{N}$$

Now the remaining 2 places with the remaining 6 letters can be filled in

${}^{6}P_{2} = 6 \times 5 = 30$ ways. Thus, the number of 4 letter words that begin with E and end with N is 30.

5.2.9 To prove ${}^{n}P_{r} = {}^{(n-1)}P_{r} + r \,.\, {}^{(n-1)}P_{(r-1)}$

In this section, we prove the following important result using the definition of ${}^{n}P_{r}$.

5.2.10 Theorem : *Let n, r be positive integers and $1 \leq r < n$. Then*

$$ {}^{n}P_{r} = {}^{(n-1)}P_{r} + r \,.\, {}^{(n-1)}P_{(r-1)}.$$

First proof (From fundamentals) : We know that ${}^{n}P_{r}$ is the number of ways of filling up r places using n things. Let us take one thing among the given n things and name it as T. Let

$$m_1 = \text{number of permutations containing T}$$

$$m_2 = \text{number of permutations not containing T.}$$

Thus, $\qquad {}^{n}P_{r} = m_1 + m_2.$

To find m_1, we first put 'T' in one of the r places. This can be done in r ways. Now we can fill the remaining $(r-1)$ places with the remaining $(n-1)$ things in ${}^{(n-1)}P_{(r-1)}$ ways. Therefore,

$$m_1 = r \,.\, {}^{(n-1)}P_{(r-1)}.$$

To find m_2, leave T aside and fill the r places with the remaining $(n-1)$ things in ${}^{(n-1)}P_{r}$ ways. Thus

$$m_2 = {}^{(n-1)}P_{r}.$$

Therefore,

$$^{n}P_{r} = m_1 + m_2 = {}^{(n-1)}P_r + r\,.\,{}^{(n-1)}P_{(r-1)}.$$

Second Proof (Using the formula of $^{n}P_{r}$)

$$^{(n-1)}P_r + r\,.\,{}^{(n-1)}P_{(r-1)} = \frac{(n-1)!}{(n-1-r)!} + r\,.\,\frac{(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!(n-r)+r.(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!.\,n}{(n-r)!} = \frac{n!}{(n-r)!} = {}^{n}P_r.$$

5.2.11 Examples

1. Example : Find the number of 4 letter words that can be formed using the letters of the word MIXTURE which

(i) contain the letter X

(ii) do not contain the letter X

Solution

We have to fill up 4 places using the 7 letters of the word MIXTURE. Take 4 places.

(i) First we put X in one of the 4 places. This can be done in 4 ways. Now we can fill the remaining 3 places with the remaining 6 letters in $^{6}P_3$ ways. Thus the number of 4 letter words containing the letter X are

$$4 \times {}^{6}P_3 = 4 \times 120 = 480$$

(ii) Leaving the letter X, we fill the 4 places with the remaining 6 letters in $^{6}P_4$ ways. Thus, the number of 4 letter words that do not contain the letter X is

$$^{6}P_4 = 360.$$

2. Example : Find the number of ways of arranging 5 boys and 4 girls in a row so that the row (i) begins and ends with boys (ii) begins with a boy and ends with a girl.

Solution : (i) The total number of persons is 9 (5 boys + 4 girls). Let us take 9 places.

B								B

First we fill the first and the last places with boys. This can be done in ${}^5P_2$ ways. Now, we have to fill up the remaining 7 places with the remaining 7 persons (3 boys + 4 girls) in 7! ways. Hence the required number of arrangements is

$$ {}^5P_2 \times 7! = 20 \times 5040 = 100800. $$

(ii)

B								G

We fill the first place with one of the boys in 5 ways and last place with one of the girls in 4 ways. The remaining 7 places can be filled with the remaining 7 persons (4 boys + 3 girls) in 7! ways. Hence the number of required arrangements is

$$ 5 \times 4 \times 7! = 20 \times 5040 = 100800. $$

5.2.12 Solved Problems

1. Problem : *If ${}^nP_4 = 1680$, find n.*

Solution : We know that ${}^nP_4$ is the product of 4 consecutive integers of which n is the largest. That is

$$ {}^nP_4 = n(n-1)(n-2)(n-3) $$

and $$ 1680 = 8 \times 7 \times 6 \times 5 $$

on comparing the largest integers, we get $n = 8$.

2. Problem : *If ${}^{12}P_r = 1320$, find r.*

Solution : $1320 = 12 \times 11 \times 10 = {}^{12}P_3$. Thus $r = 3$.

3. Problem : If ${}^{(n+1)}P_5 : {}^nP_5 = 3 : 2$, find n.

Solution : ${}^{(n+1)}P_5 : {}^nP_5 = 3 : 2$

$$ \Rightarrow \frac{(n+1)!}{(n-4)!} \times \frac{(n-5)!}{n!} = \frac{3}{2} $$

$$ \Rightarrow \frac{n+1}{n-4} = \frac{3}{2} $$

$$ \Rightarrow 2n + 2 = 3n - 12 \Rightarrow n = 14. $$

4. Problem : *If ${}^{56}P_{(r+6)} : {}^{54}P_{(r+3)} = 30800 : 1$, find r.*

Solution : ${}^{56}P_{(r+6)} : {}^{54}P_{(r+3)} = 30800 : 1$

$$ \Rightarrow \frac{(56)!}{(56-(r+6))!} \times \frac{(54-(r+3))!}{(54)!} = \frac{30800}{1} $$

$$\Rightarrow \frac{(56)!}{(50-r)!} \times \frac{(51-r)!}{(54)!} = \frac{30800}{1}$$

$$\Rightarrow 56 \times 55 \times (51 - r) = 30800$$

$$\Rightarrow (51-r) = \frac{30800}{56 \times 55} = 10$$

$$\Rightarrow r = 41.$$

5. Problem : *In how many ways 9 mathematics papers can be arranged so that the best and the worst (i) may come together (ii) may not come together ?*

Solution

(i) If the best and worst papers are treated as one unit, then we have $9 - 2 + 1 = 7 + 1 = 8$ papers. Now these can be arranged in $(7 + 1)!$ ways and the best and worst papers between themselves can be permuted in $2!$ ways. Therefore the number of arrangements in which best and worst papers come together is $8!\,2!$.

(ii) Total number of ways of arranging 9 mathematics papers is $9!$. The best and worst papers come together in $8!\,2!$ ways. Therefore the number of ways they may not come together is $9! - 8!\,2!$

$$= 8!(9 - 2) = 8! \times 7.$$

6. Problem : *Find the number of ways of arranging 6 boys and 6 girls in a row. In how many of these arrangements*

(i) all the girls are together

(ii) no two girls are together

(iii) boys and girls come alternately ?

Solution: 6 boys + 6 girls = 12 persons. They can be arranged in a row in $(12)!$ ways.

(i) Treat the 6 girls as one unit. Then we have

6 boys + 1 unit of girls.

They can be arranged in $7!$ ways. Now, the 6 girls among themselves can be permuted in $6!$ ways. Hence, by the fundamental principle, the number of arrangements in which all 6 girls are together is $7! \times 6!$.

(ii) First we arrange 6 boys in a row in 6! ways. The girls can be arranged in the 7 gaps between the boys (including the gap in the beginning and the gap in the ending). These gaps are shown below by the letter X.

X	$\boxed{B}$	X	$\boxed{B}$	X	$\boxed{B}$	X	$\boxed{B}$	X	$\boxed{B}$	X	$\boxed{B}$	X
1		2		3		4		5		6		7

Now, the 6 girls can be arranged in these 7 gaps in ${}^7P_6$ ways. Hence, by the fundamental principle, the number of arrangements in which no two girls come together is $6! \times {}^7P_6 = 6! \times 7! = 7 \times 6! \times 6!$

(iii) Let us take 12 places. The row may begin with either a boy or a girl. That is, 2 ways. If it begins with a boy, then all odd places (1, 3, 5, 7, 9, 11) will be occupied by boys and the even places (2, 4, 6, 8, 10, 12) by girls. The 6 boys can be arranged in the 6 odd places in 6! ways and the 6 girls can be arranged in the 6 even places in 6! ways. Thus the number of arrangements in which boys and girls come alternately is $2 \times 6! \times 6!$.

Note

In the above, one may think that questions (ii) and (iii) are same. But they are not (as evident from the answers). In Question (ii), after arranging 6 boys, we found 7 gaps and 6 girls are arranged in these 7 gaps. Hence one place remains vacant. It can be any one of the 7 gaps. But in Question (iii), the vacant place should either be at the beginning or at the ending but not in between. Thus, only 2 choices for the vacant place

7. Problem. *Find the number of 4-letter words that can be formed using the letters of the word MIRACLE. How many of them*

(i) begin with an vowel

(ii) begin and end with vowels

(iii) end with a consonant?

Solution : The word MIRACLE has 7 letters. Hence the number of 4 letter words that can be formed using these letters is ${}^7P_4 = 7 \times 6 \times 5 \times 4 = 840$

Let us take 4 blanks.

$\boxed{}\quad\boxed{}\quad\boxed{}\quad\boxed{}$

(i) We can fill the first place with one of the 3 vowels (I, A, E) in ${}^3P_1 = 3$ ways. Now, the remaining 3 places can be filled using the remaining 6 letters in

$${}^6P_3 = 120 \text{ ways.}$$

Thus the number of 4 letter words that begin with an vowel is $3 \times 120 = 360$.

(ii) Fill the first and last places with 2 vowels in ${}^{3}P_{2} = 6$ ways.

The remaining 2 places can be filled with the remaining 5 letters in ${}^{5}P_{2} = 20$ ways.

Thus the number of 4 letter words that begin and end with vowels is

$$6 \times 20 = 120.$$

(iii) We can fill the last place with one of the 4 consonants (M, R, C, L) in ${}^{4}P_{1} = 4$ ways.

The remaining 3 places can be filled with the remaining 6 letters in ${}^{6}P_{3}$ ways.

Thus the number of 4 letter words that end with a consonant

$$4 \times {}^{6}P_{3} = 4 \times 120 = 480.$$

8. Problem : *Find the number of ways of permuting the letters of the word PICTURE so that*

(i) *all vowels come together*

(ii) *no two vowels come together.*

(iii) *the relative positions of vowels and consonants are not distrurbed.*

Solution : The word PICTURE has 3 vowels (I, U, E) and 4 consonants (P, C, T, R)

(i) Treat the 3 vowels as one unit. Then we can arrange 4 consonants + 1 unit of vowels in 5! ways. Now the 3 vowels among themselves can be permuted in 3! ways. Hence the number of permutations in which the 3 vowels come together is $5! \times 3! = 720$.

(ii) First arrange the 4 consonants in 4! ways. Then in between the vowels, in the beginning and in the ending, there are 5 gaps as shown below by the letter X

X	□	X	□	X	□	X	□	X
1		2		3		4		5

In these 5 places we can arrange the 3 vowels in ${}^{5}P_{3}$ ways. Thus the number of words in which no two vowels come together is $4! \times {}^{5}P_{3} = 24 \times 60 = 1440$.

(iii) The three vowels can be arranged in their relative positions in 3! ways and the 4 consonants can be arranged in their relative positions in 4! ways.

V	C	C	V	C	V	C

The required number of arrangements is $3!\,4! = 144$.

Note

In the above problem, from (i) we get that the number of permutations in which the vowels do not come together is

= Total number of permutations – number of permutations in which 3 vowels come together.

$= 7! - 5!\,.\,3! = 5040 - 720 = 4320.$

But the number of permutations in which no two vowels come together is only 1440. In the remaining 2880 permutations two vowels come together and third appears away from these.

9. Problem : *If the letters of the word PRISON are permuted in all possible ways and the words thus formed are arranged in dictionary order, find the rank of the word PRISON.*

Solution : The letters of the given word in dictionary order is

I N O P R S

In the dictionary order, first we write all words that begin with I. If we fill the first place with I, then the remaining 5 places can be filled with the remaining 5 letters in 5! ways. That is, there are 5! words that begin with I. Proceeding like this, after writing all words that begin with I, N, O, we have to write the words that begin with P. Among them first come the words with first two letters P, I. As above there are 4! such words. On proceeding like this, we get

I	–	–	–	–	–	$\rightarrow$	5! words
N	–	–	–	–	–	$\rightarrow$	5! words
O	–	–	–	–	–	$\rightarrow$	5! words
P	I	–	–	–	–	$\rightarrow$	4! words
P	N	–	–	–	–	$\rightarrow$	4! words
P	O	–	–	–	–	$\rightarrow$	4! words
P	R	I	N	–	–	$\rightarrow$	2! words
P	R	I	O	–	–	$\rightarrow$	2! words
P	R	I	S	N	–	$\rightarrow$	1! words
P	R	I	S	O	N	$\rightarrow$	1 word

Hence the rank of the word PRISON is

$$3 \times 5! + 3 \times 4! + 2 \times 2! + 1! + 1 = 360 + 72 + 4 + 1 + 1 = 438.$$

10.Problem : *Find the number of 4-digit numbers that can be formed using the digits 2, 3, 5, 6, 8 (without repetition). How many of them are divisible by*

(i) 2 (ii) 3 (iii) 4 (iv) 5 (v) 25

Solution

The number of 4-digit numbers that can be formed using the 5 digits 2, 3, 5, 6, 8 is $^5P_4 = 120$.

(i) Divisible by 2 : For a number to be divisible by 2, the units place should be filled with an even digit. This can be done in 3 ways (2 or 6 or 8).

			X

Now, the remaining 3 places can be filled with the remaining 4 digits in $^4P_3 = 24$ ways. Hence, the number of 4-digit numbers divisible by 2 is

$$3 \times 24 = 72.$$

(ii) Divisible by 3 : A number is divisible by 3 if the sum of the digits in it is a multiple of 3. Since the sum of the given 5 digits is 24, we have to leave either 3 or 6 and use the digits 2, 5, 6, 8 or 2, 3, 5, 8. In each case, we can permute them in 4! ways. Thus the number of 4-digit numbers divisible by 3 is

$$2 \times 4! = 48.$$

(iii) Divisible by 4 : A number is divisible by 4 if the number formed by the digits in the last two places (tens and units places) is a multiple of 4.

		X	X

Thus we fill the last two places (as shown in the figure) with one of

28, 32, 36, 52, 56, 68

That is done in 6 ways. After filling the last two places, we can fill the remaining two places with the remaining 3 digits in $^3P_2 = 6$ ways.

Thus, the number of 4-digit numbers divisible by 4 is $6 \times 6 = 36$.

(iv) Divisible by 5 : After filling the units place with 5 (one way), the remaining 3 places can be filled with the remaining 4 digits in $^4P_3 = 24$ ways. Hence the number of 4-digit numbers divisible by 5 is 24.

(v) Divisible by 25 : Here also we have to fill the last two places (that is, units and tens place) with 25 (one way) as shown below.

		2	5

Now the remaining 2 places can be filled with the remaining 3 digits in $^3P_2 = 6$ ways. Hence the number of 4-digit numbers divisible by 25 is 6.

11. Problem : *Find the sum of all 4-digit numbers that can be formed using the digits 1, 3, 5, 7, 9.*

Solution : We know that the number of 4-digit numbers that can be formed using the given 5 digits is ${}^5P_4 = 120$. Now we find their sum.

We first find the sum of the digits in the unit place of all these 120 numbers. If we fill the units place with 1 as shown below,

			1

then the remaining 3 places can be filled with the remaining 4 digits in ${}^4P_3$ ways. This means, the number of 4 digit numbers having 1 in units place is ${}^4P_3$. Similarly, each of the digits 3, 5, 7, 9 appear 24 times in units place. By adding all these digits we get the sum of the digits in units place of all 120 numbers as

$${}^4P_3 \times 1 + {}^4P_3 \times 3 + {}^4P_3 \times 5 + {}^4P_3 \times 7 + {}^4P_3 \times 9 = {}^4P_3 \times 25.$$

Similarly, we get the sum of the digits in Tens place as ${}^4P_3 \times 25$. Since it is in 10's place, its value is ${}^4P_3 \times 25 \times 10$.

Similarly, the values of the sum of the digits in 100's place and 1000's place are

$${}^4P_3 \times 25 \times 100 \text{ and } {}^4P_3 \times 25 \times 1000$$

respectively. Hence the sum of all the 4-digit numbers formed by using the digits 1, 3, 5, 7, 9 is

$$\begin{aligned} &{}^4P_3 \times 25 \times 1 + {}^4P_3 \times 25 \times 10 + {}^4P_3 \times 25 \times 100 + {}^4P_3 \times 25 \times 1000 \\ &= {}^4P_3 \times 25 \times 1111 \qquad \ldots (*) \\ &= 24 \times 25 \times 1111 = 6{,}66{,}600. \end{aligned}$$

Note

1. From (*) in the above example, we can derive that the sum of all r-digit numbers that can be formed using the given 'n' non - zero digits ($1 \le r \le n \le 9$) is

$${}^{(n-1)}P_{(r-1)} \times \text{sum of the given digits} \times 111..1 \ (r \text{ times})$$

2. In the above, if '0' is one digit among the given n digits, then we get that the sum of the r-digit numbers that can be formed using the given n digits (including '0')

$$\begin{aligned} &= \{{}^{(n-1)}P_{(r-1)} \times \text{sum of the given digits} \times 111..1 \ (r \text{ times}) \} \\ &\quad - \{{}^{(n-2)}P_{(r-2)} \times \text{sum of the given digits} \times 111..1 \ ((r-1) \text{ times})\}. \end{aligned}$$

12. Problem : *How many four digited numbers can be formed using the digits 1, 2, 5, 7, 8, 9? How many of them begin with 9 and end with 2 ?*

Solution : The number of four digited numbers that can be formed using the given digits 1, 2, 5, 7, 8, 9 is ${}^6P_4 = 360$. Now, the first place and last place can be filled with 9 and 2 in one way.

9			2

The remaining 2 places can be filled by the remaining 4 digits 1, 5, 7, 8. Therefore these two places can be filled in ${}^4P_2$ ways. Hence, the required number of ways = $1\,.\,{}^4P_2 = 12$.

13. Problem : *Find the number of injections of a set A with 5 elements to a set B with 7 elements.*

Solution : The first element of A can be mapped to any one of the 7 elements in 7 ways. The second element of A can be mapped to any one of the remaining 6 elements in 6 ways. Proceeding like this we get the number of injections from A to B as ${}^7P_5 = 2520$.

Note

> If a set A has m elements and the set B has n elements, then the number of injections from A into B is ${}^nP_m$ if $m \le n$ and 0 if $m > n$.

14. Problem : *Find the number of ways in which 4 letters can be put in 4 addressed envelopes so that no letter goes into the envelope meant for it.*

Solution : Required number of ways is $4!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right) = 12 - 4 + 1 = 9$.

Note

> If there are n things in a row, a permutation of these n things such that none of them occupies its original position is called a derangement of n things.
>
> The number of derangements of n distinct things is $n!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots+(-1)^n\frac{1}{n!}\right)$.

Exercise 5(a)

I. 1. If ${}^nP_3 = 1320$, find n.

2. If ${}^nP_7 = 42\,.\,{}^nP_5$, find n.

3. If ${}^{(n+1)}P_5 : {}^nP_6 = 2 : 7$, find n.

4. If ${}^{12}P_5 + 5\,.\,{}^{12}P_4 = {}^{13}P_r$, find r.

5. If ${}^{18}P_{(r-1)} : {}^{17}P_{(r-1)} = 9 : 7$, find r.

6. A man has 4 sons and there are 5 schools within his reach. In how many ways can he admit his sons in the schools so that no two of them will be in the same school.

II. 1. If there are 25 railway stations on a railway line, how many types of single second class tickets must be printed, so as to enable a passenger to travel from one station to another.

2. In a class there are 30 students. On the New year day, every student posts a greeting card to all his / her classmates. Find the total number of greeting cards posted by them.

3. Find the number of ways of arranging the letters of the word TRIANGLE so that the relative positions of the vowels and consonants are not disturbed.

4. Find the sum of all 4 digited numbers that can be formed using the digits 0, 2, 4, 7, 8, without repetition.

5. Find the number of numbers that are greater than 4000 which can be formed using the digits 0, 2, 4, 6, 8 without repetition.

6. Find the number of ways of arranging the letters of the word MONDAY so that no vowel occupies even place.

7. Find the number of ways of arranging 5 different mathematics books, 4 different physics books and 3 different chemistry books such that the books of the same subject are together.

III. 1. Find the number of 5 letter words that can be formed using the letters of the word CONSIDER. How many of them begin with "C", how many of them end with "R", and how many of them begin with "C" and end with "R" ?

2. Find the number of ways of arranging 10 students $A_1, A_2, ..., A_{10}$ in a row such that

(i) A_1, A_2, A_3 sit together

(ii) A_1, A_2, A_3 sit in a specified order.

(iii) A_1, A_2, A_3 sit together in a specified order.

3. Find the number of ways in which 5 red balls, 4 black balls of different sizes can be arranged in a row so that (i) no two balls of the same colour come together (ii) the balls of the same colour come together.

4. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 5, 6, 7. How many of them are divisible by

(i) 2 (ii) 3 (iii) 4 (iv) 5 (v) 25

5. If the letters of the word MASTER are permuted in all possible ways and the words thus formed are arranged in the dictionary order, then find the ranks of the words (i) REMAST (ii) MASTER.

6. If the letters of the word BRING are permuted in all possible ways and the words thus formed are arranged in the dictionary order, then find the 59th word.

7. Find the sum of all 4 digited numbers that can be formed using the digits 1, 2, 4, 5, 6 without repetition.

8. There are 9 objects and 9 boxes. Out of 9 objects, 5 cannot fit in three small boxes. How many arrangements can be made such that each object can be put in one box only.

5.3 Permutations when repetitions are allowed

In the previous sections we have learnt about the number of permutations of 'n' dissimilar things taken 'r' at a time, when the repetition of the things is not allowed. In this section, we learn about the number of permutations of n dissimilar things taken r at a time when each thing can be repeated any number of times. (That is, when repetition is allowed).

5.3.1 Theorem : Let n and r be positive integers. If the repetition of things is allowed, then the number of permutations of 'n' dissimilar things taken 'r' at a time is n^r.

Proof : The number of required permutations is equal to the number of ways of filling up 'r' blank places with the given n things (repetitions allowed). We prove this by using induction on r. If $r = 1$, then the number of ways of filling up one blank using the given n things is $n = n^1$. Therefore the result is true for $r = 1$. Assume that $r > 1$ and that the result is true for $(r-1)$. That is the number of ways of filling up $(r-1)$ blank places with the given n things is $n^{(r-1)}$.Now suppose 'n' dissimilar things are given. Now we take r blank places as shown below.

$$\underset{1}{\square} \quad \underset{2}{\square} \quad \underset{3}{\square} \quad \cdots\cdots \quad \underset{r}{\square}$$

The blank 1 can be filled with any one of the given n things in 'n' ways. Now, we are left with $(r-1)$ blanks and n things (because the object used in the first place can be used again). By induction hypothesis the remaining $(r-1)$ places can be filled with the given n things in $n^{(r-1)}$ ways. Therefore, by the fundamental principle, the number of ways of filling up 'r' blanks with the given 'n' things is

$$n \times n^{(r-1)} = n^r.$$

Hence the theorem follows by mathematical induction.

Note : The number of permutations of n dissimilar things taken 'r' things at a time with atleast one repetition is $n^r - {}^nP_r$.

5.3.2 Definition

A number or a word which reads same either from left to right or from right to left is called a Palindrome. Some examples of palindromes are ATTA, ROTOR, 12321, 120021 etc.

Note : The number of palindromes with r distinct letters that can be formed using given n distinct letters is

(i) $n^{r/2}$ if r is even

(ii) $n^{\frac{r+1}{2}}$ if r is odd.

5.3.3 Examples

1. Example : Find the number of permutations of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5, 6 when repetition is allowed.

Solution : From Theorem 5.3.1, the number of 4-digit numbers is

$$6^4 = 1296.$$

2. Example : Find the number of 4 letter words that can be formed using the letters of the word PISTON in which atleast one letter is repeated.

Solution : The given word has 6 letters. The number of 4 letter words that can be formed using these 6 letters

(i) when repetition is allowed is 6^4

(ii) When repetition is not allowed is ${}^6P_4$

Hence, the number 4 letter words in which atleast one letter is repeated is

$$6^4 - {}^6P_4 = 1296 - 360 = 936.$$

3. Example : A number lock has 3 rings and each ring has 9 digits 1, 2, 3, ..., 9. Find the maximum number of unsuccessful attempts that can be made by a person who tries to open the lock without knowing the key code.

Solution : Each ring can be rotated in 9 different ways. Thus, the total number of different ways in which 3 rings can be rotated is 9^3. Out of these attempts, only one attempt is a successful attempt. Hence, the maximum number of unsuccessful attempts is

$$9^3 - 1 = 729 - 1 = 728.$$

4. Example : Find the number of (i) 6 (ii) 7 letter palindromes that can be formed using the letters of the word EQUATION.

Solution : (i) 8^3 (ii) 8^4 (using the note below 5.3.2 definition)

5. Example : Find the number of seven digited palindromes that can be formed using 0, 1, 2, 3, 4.

Solution : First place can be filled in 4 ways (using only non-zero digits). Remaining three places can be filled in 5 ways each.

$\therefore$ Number of palindromes $= 4 \times 5^3$.

5.3.4 Solved Problems

1. Problem : *Find the number of 5-letter words that can be formed using the letters of the word MIXTURE which begin with an vowel when repetitions are allowed.*

Solution : We have to fill up 5 blanks using the letters of the word MIXTURE having 7 letters among which there are 3 vowels. Fill the first place with one of the vowels (I or U or E) in 3 ways as shown below

X				

Each of the remaining 4 places can be filled in 7 ways(since we can use all 7 letters each time). Thus the number of 5-letter words is $3 \times 7 \times 7 \times 7 \times 7 = 3 \times 7^4$.

2. Problem: *Find the number of functions from a set A with m elements to a set B with n elements.*

Solution : Let A = $\{a_1, a_2, ..., a_m\}$ and B = $\{b_1, b_2, ..., b_n\}$

To define the image of a_1 we have n choices (any element of B). Then we can define the image of a_2 again in n ways (since a_1, a_2 can have same image). Thus we can define the image of each of the m elements in n ways. Therefore the number of functions from A to B is

$$n \times n \times ... \times n \ (m \text{ times}) = n^m.$$

3. Problem : *Find the number of surjections from a set A with n elements to a set B with 2 elements when $n>1$.*

Solution : Let A = $\{a_1\ a_2 ... a_n\}$ and B = $\{x, y\}$. From the above problem, the total number of functions from A onto B is 2^n. For a function to be a surjection its range should contain both x, y. Observe that the number of functions which are not surjections that is, the functions which contain x or y alone in the range is 2. Hence the number of surjections from A to B is $2^n - 2$.

Note : In the above problem, even if B has more than 2 elements also we can derive a formula to find the number of surjections from A to B. But this result is beyond the scope of this book and hence it is not included here.

4. Problem : *Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5, 6 that are divisible by*

(i) 2 (ii) 3

when repetition is allowed.

Solution

(i) Numbers divisible by 2

Take 4 blanks. For a number to be divisible by 2, the units place should be filled with an even digit. This can be done in 3 ways (2 or 4 or 6).

			X

Now, each of the remaining 3 places can be filled in 6 ways. Hence the number of 4-digit numbers that are divisible by 2 is

$$3 \times 6^3 = 3 \times 216 = 648.$$

(ii) Numbers divisible by 3

Fill the first 3 places with the given 6 digits in 6^3 ways.

X	X	X	

Now, after filling up the first 3 places with three digits, if we fill up the units place in 6 ways, we get 6 consecutive positive integers. Out of any six consecutive integers exactly **two** are divisible by 3. Hence the units place can be filled in 2 ways. Hence the number of 4-digit numbers divisible by 3 is $2 \times 216 = 432$.

5. Problem : *Find the number of 5-letter words that can be formed using the letters of the word EXPLAIN that begin and end with a vowel when repetitions are allowed.*

Solution : We can fill the first and last places with vowels each in 3 ways (E or A or I).

X				X

Now each of the remaining 3 places can be filled in 7 ways (using any letter of given 7 letters). Hence the number of 5 letter words which begin and end with vowels is

$$3^2 \times 7^3 = 9 \times 343 = 3087.$$

Exercise 5(b)

I. 1. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 4, 5, 7, 8 when repetition is allowed.

2. Find the number of 5 letter words that can be formed using the letters of the word RHYME if each letter can be used any number of times.

3. Find the number of functions from a set A containing 5 elements into a set B containing 4 elements.

II. 1. Find the number of palindromes with 6 digits that can be formed using the digits

(i) 0, 2, 4, 6, 8 (ii) 1, 3, 5, 7, 9

2. Find the number of 4-digit telephone numbers that can be formed using the digits 1, 2, 3, 4, 5, 6 with atleast one digit repeated.

3. Find the number of bijections from a set A containing 7 elements onto itself.

4. Find the number of ways of arranging 'r' things in a line using the given 'n' different things in which atleast one thing is repeated.

5. Find the number of 5 letter words that can be formed using the letters of the word NATURE that begin with N when repetition is allowed.

6. Find the number of 5-digit numbers divisible by 5 that can be formed using the digits 0, 1, 2, 3, 4, 5, when repetition is allowed.

7. Find the number of numbers less than 2000 that can be formed using the digits, 1, 2, 3, 4 if repetition is allowed.

III. 1. 9 different letters of an alphabet are given. Find the number of 4 letter words that can be formed using these 9 letters which have

(i) no letter is repeated (ii) atleast one letter is repeated.

2. Find the number of 4-digit numbers which can be formed using the digits 0, 2, 5, 7, 8 that are divisible by (i) 2 (ii) 4 when repetition is allowed.

3. Find the number of 4-digit numbers that can be formed using the digits 0, 1, 2, 3, 4, 5 which are divisible by 6 when repetition of the digits is allowed.

5.4 Circular Permutations

In all the previous sections in this chapter, we have studied about linear permutations (That is, permutations arranged linearly) with or without repetition. In this section, we learn about the arrangement of given objects around a circle. These permutations are called **Circular permutations** (Definition 5.1.6).

In circular permutations, there are two types of arrangements. One is clock-wise and the other is anti clock-wise as shown in the Figures 5.1 and 5.2

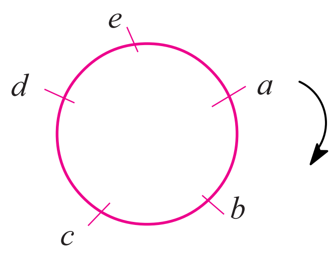


Fig. 5.1
Clock - wise arrangement

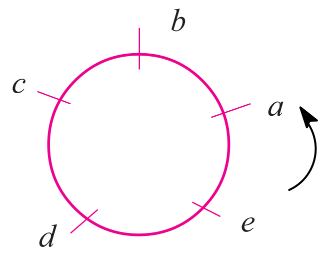


Fig. 5.2
Anti-clock-wise arrangement

The above two circular permutations are same but for the direction. In circular permutations, in general, the direction is also important and hence we regard the two permutations described in the Fig. 5.1 and Fig. 5.2 as two different circular permutations. However, in some special cases we treat the clock - wise and anti clock-wise arrangements of the same circular permutations as identical. We will discuss such cases later.

5.4.1 Theorem

The number of circular permutations of 'n' dissimilar things (taken all at a time) is $(n-1)!$

Proof : First method

In a circular permutation, there is no first place or beginning place. Hence which thing we use first or which place we fill first does not matter. But how we arrange the remaining things relative to the first object already placed is to be calculated. Take n places around a circle as shown in the Fig. 5.3.

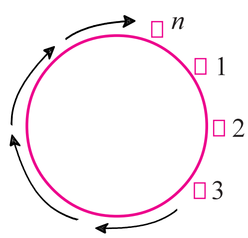


Fig. 5.3

Put any one of the given n things in any one of the n places. Now the remaining $(n-1)$ things can be arranged in the remaining $(n-1)$ places in $(n-1)!$ ways. Therefore, the number of circular permutations of n things taken all at a time is $(n-1)!$

Second Method

Let M be the number of circular permutations of n things taken all at a time. If we take one such permutation it looks as in the Fig. 5.4.

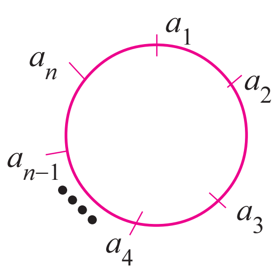


Fig. 5.4

This circular permutation gives rise to n linear permutations (read in either clock-wise or anti clock - wise direction, but not both) as exhibited below.

$$a_1\, a_2\, a_3 \,....\, a_{n-1}\, a_n$$

$$a_2\, a_3\, a_4 \,....\, a_n\, a_1$$

$$a_3\, a_4\, a_5 \,...\, a_1\, a_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_n\, a_1\, a_2 \,...\, a_{n-2}\, a_{n-1}$$

Thus, one circular permutation gives rise to n linear permutations and hence M circular permutations give us $M \times n$ linear permutations. But we know that the number of linear permutations of n dissimilar things (taken all at a time) is ${}^{n}P_{n} = n!$

Hence, we get $M \times n = n!$ and $M = \dfrac{n!}{n} = (n-1)!$.

5.4.2 Note

In case of (hanging type of circular permutations like) the garlands of flowers, chains of beads etc. a circular permutation looks like clock-wise arrangement when seen from one side and anti clock-wise arrangement from the other side. Hence, they will be treated as identical. Therefore, the number of circular permutations, of n things in these cases is $\dfrac{(n-1)!}{2}$ (half of the number of the actual circular permutations).

5.4.3 Example : Find the number of ways of arranging 5 boys and 5 girls around a circle.

Solution : Total number of persons $n = 10$ (5 boys + 5 girls).

Therefore, the number of circular permutations is $(n-1)! = 9!$

5.4.4 Example : Find the number of ways of arranging 8 persons around a circular table if two particular persons were to sit together.

Solution : Treat the two particular persons as one unit. Then we have 6 + 1 = 7 entities. They can be arranged around a circular table in 6! ways. Now, the two particular persons can be permuted among themselves in 2! ways. Therefore, the number of required arrangements is

$$6! \times 2! = 1440.$$

5.4.5 Solved Problems

1. Problem : *Find the number of ways of arranging 8 men and 4 women around a circular table. In how many of them*

(i) *all the women come together*

(ii) *no two women come together*

Solution : Total number of persons = 12 (8 men + 4 women)

Therefore, the number of circular permutations is (11)!

(i) Treat the 4 women as single unit. Then we have

8 men + 1 unit of women = 9 entities.

They can be arranged around a circular table in 8! ways. Now, the 4 women among themselves can be arranged in 4! ways. Hence by the Fundamental principle, the required number of arrangements is $8! \times 4!$

(ii) First we arrange 8 men around the circular table in 7! ways. There are 8 places in between them as shown in Fig. 5.5 by the symbol x. (one place in between any two consecutive men)

Now we can arrange the 4 women in these 8 places in ${}^8P_4$ ways. Thus, the number of circular permutations in which no two women come together is

$$7! \times {}^8P_4.$$

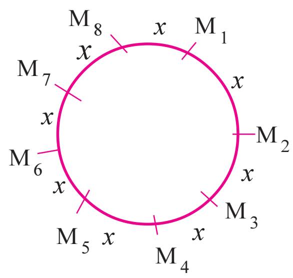


Fig. 5.5

2. Problem : *Find the number of ways of arranging 5 Indians, 4 Americans and 3 Russians at a round table so that*

(i) *all Indians sit together*

(ii) *no two Russians sit together*

(iii) *persons of same nationality sit together.*

Solution

(i) Treat the 5 Indians as one unit. Then we have 4 Americans + 3 Russians + 1 unit of Indians = 8 entities.

They can be arranged at a round table in $(8 - 1)! = 7!$ ways.

Now, the 5 Indians among themselves can be arranged in 5! ways. Hence, the required number of arrangements is $7! \times 5!$.

(ii) First we arrange the 5 Indians + 4 Amercians around the table in $(9-1)! = 8!$ ways.

Now, there are 9 gaps in between these 9 persons (one gap between any two consecutive persons). The 3 Russians can be arranged in these 9 gaps in ${}^{9}P_{3}$ ways. Hence, the required number of arrangements is

$$8! \times {}^{9}P_{3}$$

(iii) Treat the 5 Indians as one unit, the 4 Amercians as one unit and the 3 Russians as one unit. These 3 units can be arranged at round table in $(3-1)! = 2!$ ways.

Now, the 5 Indians among themselves can be permuted in 5! ways. Similarly, the 4 Amercians in 4! ways and 3 Russians in 3! ways. Hence, the required number of arrangements is

$$2! \times 5! \times 4! \times 3!$$

3. Problem : *Find the number of different chains that can be prepared using 7 different coloured beads.*

Solution : We know that the number of circular permutations of hanging type that can be formed using n things is $\frac{1}{2}\{(n-1)!\}$. Hence the number of different ways of preparing the chains $= \frac{1}{2}\{(7-1)!\} = \frac{6!}{2} = 360$.

4. Problem : *Find the number of different ways of preparing a garland using 7 distinct red roses and 4 distinct yellow roses such that no two yellow roses come together.*

Solution : First we arrange 7 red roses in a circular form (garland form) in $(7-1)! = 6!$ ways Now, there are 7 gaps in between the red roses and we can arrange the 4 yellow roses in these 7 gaps in ${}^{7}P_{4}$ ways. Thus the total number of circular permutations is $6! \times {}^{7}P_{4}$.

But, this being the case of garlands, clock-wise and anti clock-wise arrangements look a like. Hence, the required number of ways is $\frac{1}{2}\left(6! \times {}^{7}P_{4}\right)$

Exercise 5(c)

I. 1. Find the number of ways of arranging 7 persons around a circle.

2. Find the number of ways of arranging the chief minister and 10 cabinet ministers at a circular table so that the chief minister always sits in a particular seat.

3. Find the number of ways of preparing a chain with 6 different coloured beads.

II. 1. Find the number of ways of arranging 4 boys and 3 girls around a circle so that all the girls sit together.

2. Find the number of ways of arranging 7 gents and 4 ladies around a circular table if no two ladies wish to sit together.

3. Find the number of ways of arranging 7 guests and a host around a circle if 2 particular guests wish to sit on either side of the host.

4. Find the number of ways of preparing a garland with 3 yellow, 4 white and 2 red roses of different sizes such that the two red roses come together.

III. 1. Find the number of ways of arranging 6 boys and 6 girls around a circular table so that

(i) all the girls sit together (ii) no two girls sit together

(iii) boys and girls sit alternately

2. Find the number of ways of arranging 6 red roses and 3 yellow roses of different sizes into a garland. In how many of them

(i) all the yellow roses are together (ii) no two yellow roses are together

3. A round table conference is attended by 3 Indians, 3 Chinese, 3 Canadians and 2 Americans. Find the number of ways of arranging them at the round table so that the delegates belonging to same country sit together.

4. A chain of beads is to be prepared using 6 different red coloured beads and 3 different blue coloured beads. In how many ways can this be done so that no two blue coloured beads come together.

5. A family consists of father, mother, 2 daughters and 2 sons. In how many different ways can they sit at a round table if the 2 daughters wish to sit on either side of the father?

5.5 Permutations with Constraint repetitions

In section 5.3, we have learnt about the permutations in which repetition of things is allowed. That is, each of the given n dissimilar things can be used any number of times. But in the literature we come across many words whose spelling contains certain repeated letters like MATHEMATICS, COFFEE, ASSOCIATION, EAMCET etc. and many numbers with repeated digits like 47436, 3007, 141516 etc. In this section, we find the number of permutations of such words or numbers.

5.5.1 Theorem

The number of linear permutations of 'n' things in which 'p' things are alike and the rest are different is $\dfrac{n!}{p!}$.

Proof : Let M be the number of permutations of 'n' things in which 'p' things are alike and the rest are different. If we take one such permutation, it contains p things which are alike. If we replace these p like things by p dissimilar things, then we can arange among themselves (without disturbing the relative positions of the other things) in $p!$ ways. In other words, one permutation when p things are alike gives rise to $p!$ permutations when all are different. Therefore, from the M such permutations, we get $M \times (p!)$ permutations when all n things are different. But, we know that the number of permutations of n dissimilar things (taken all at a time) is $n!$. Hence

$$M \times p! = n! \quad \text{or} \quad M = \frac{n!}{p!}$$

We can extend this result for the case when we have more than one set of alike things in the given n things by applying theorem 5.5.1 repeatedly as given in the following.

5.5.2 Theorem

The number of linear permutations of 'n' things in which there are p like things of one kind, q like things of second kind, r like things of the third kind and the rest are different is

$$\frac{n!}{p!\,q!\,r!}$$

Now, we apply these results in the following examples.

5.5.3 Example : Find the number of ways of arranging the letters of the word SPECIFIC. In how many of them

(i) the two C's come together?

(ii) the two I's do not come together?

Solution : The given word has 8 letters in which there are 2 I's (alike letters) and 2 C's. Hence, they can be arranged in

$$\frac{8!}{2!\,2!} = 10{,}080 \text{ ways (using the Theorem 5.5.2)}$$

(i) Treat the 2 C's as one unit. Then we have 6 + 1 = 7 letters in which two letters (I's) are alike. Hence, by theorem 5.5.1 they can be arranged in

$$\frac{7!}{2!} = 2520 \text{ ways}$$

Now, the 2C's among themselves can be arranged in $\frac{2!}{2!} = 1$ way. Thus, the number of required arrangements is 2520.

(ii) Keeping the 2 I's aside, arrange the remaining 6 letters can be arranged in $\frac{6!}{2!}$ = 360 ways (since there are two C's among 6 letters). Among these 6 letters we find 7 gaps as shown below

$$- S - P - E - C - F - C -$$

The two I's can be arranged in these 7 gaps in

$$\frac{{}^{7}P_{2}}{2!} \text{ ways}$$

Hence, the number of required arrangements is

$$\frac{6!}{2!} \times \frac{{}^{7}P_{2}}{2!} = 360 \times 21 = 7560.$$

5.5.4 Solved Problems

1. Problem : *Find the number of ways of arranging the letters of the word SINGING so that*

(i) *they begin and end with I*

(ii) *the two G's come together*

(iii) *relative positions of vowels and consonants are not disturbed.*

Solution

(i) First we fill the first and last places with I's in $\frac{2!}{2!}$ = 1 way as shown below

I						I

Now, we fill the remaining 5 places with the remaining 5 letters S, N, G, N, G in

$$\frac{5!}{2!\,2!} = 30 \text{ ways.}$$

Hence, the number of required permutations is 30.

(ii) Treat the two G's as one unit. Then we have 6 letters in which there are 2 I's and 2 N's. Hence they can be arranged in

$$\frac{6!}{2!\,2!} = 180 \text{ ways.}$$

Now, the two G's among themselves can be arranged in $\frac{2!}{2!}$ = 1 way. Hence the number of required permutations is 180.

(iii) In the word SINGING, there are 2 vowels which are alike i.e., I, and there are 5 consonants of which 2N's and 2G's are alike and one S is different.

$$\underline{C}\ \ \underline{V}\ \ \underline{C}\ \ \underline{C}\ \ \underline{V}\ \ \underline{C}\ \ \underline{C}$$

The two vowels can be interchanged among themselves in $\frac{2!}{2!} = 1$ way. Now, the 5 consonants can be arranged in the remaining 5 places in $\frac{5!}{2!2!} = 30$ ways.

$\therefore$ Number of required arrangements $= 1 \times 30 = 30$.

2. Problem : *Find the number of ways of arranging the letters of the word $a^4\, b^3\, c^5$ in its expanded form.*

Solution : The expanded form of $a^4\, b^3\, c^5$ is

$$aaaa\ \ bbb\ \ ccccc$$

This word has 12 letters in which there are 4 a's, 3 b's and 5c's. By Theorem 5.5.2, they can be arranged in $\frac{12!}{4!\ 3!\ 5!}$ ways.

3. Problem : *Find the number of 5-digit numbers that can be formed using the digits 1, 1, 2, 2, 3. How many of them are even?*

Solution : In the given 5 digits, there are two 1's and two 2's. Hence they can be arranged in

$$\frac{5!}{2!\ 2!} = 30 \text{ ways.}$$

Now, to find even numbers fill the units place by 2. Now the remaining 4 places can be filled using the remaining digits 1, 1, 2, 3, in

$$\frac{4!}{2!} = 12 \text{ ways}$$

Thus the number of 5-digit even numbers that can be formed using the digits 1, 1, 2, 2, 3 is 12.

4. Problem : *There are 4 copies (alike) each of 3 different books. Find the number of ways of arranging these 12 books in a shelf in a single row.*

Solution : We have 12 books in which 4 books are alike of one kind, 4 books are alike of second kind and 4 books are alike of third kind. Hence, by Theorem 5.5.2., they can be arranged in a shelf in a row in

$$\frac{12!}{4!\ 4!\ 4!} \text{ ways}$$

In problem 9 of solved problems 5.2.12, we have calculated the rank of the word PRISION. In the following problem we find the rank of a word when it contains repeated letters.

5. Problem : *If the letters of the word EAMCET are permuted in all possible ways and if the words thus formed are arranged in the dictionary order, find the rank of the word EAMCET.*

Solution : The dictionary order of the letters of given word is

A C E E M T

In the dictionary order the words which begin with the letter A come first. If we fill the first place with A, remaining 5 letters can be arranged $\frac{5!}{2!}$ ways (since there are two E's). On proceeding like this (as in problem 9 of 5.2.12) we get

$$A\ -\ -\ -\ -\ - \longrightarrow \frac{5!}{2!} \text{ words}$$

$$C\ -\ -\ -\ -\ - \longrightarrow \frac{5!}{2!} \text{ words}$$

$$E\ A\ C\ -\ -\ - \longrightarrow 3! \text{ words}$$

$$E\ A\ E\ -\ -\ - \longrightarrow 3! \text{ words}$$

$$E\ A\ M\ C\ E\ T \longrightarrow 1 \text{ word}$$

Hence the rank of the word EAMCET is

$$2 \times \frac{5!}{2!} + 2 \times 3! + 1 = 120 + 12 + 1 = 133.$$

Exercise 5(d)

I. 1. Find the number of ways of arranging the letters of the words.

(i) INDEPENDENCE (ii) MATHEMATICS

(iii) SINGING (iv) PERMUTATION

(v) COMBINATION (vi) INTERMEDIATE

2. Find the number of 7-digit numbers that can be formed using the digits 2, 2, 2, 3, 3, 4, 4.

II. 1. Find the number of 4-letter words that can be formed using the letters of the word RAMANA.

2. How many numbers can be formed using all the digits 1, 2, 3, 4, 3, 2, 1 such that even digits always occupy even places?

3. In a library, there are 6 copies of one book, 4 copies each of two different books, 5 copies each of three different books and 3 copies each of two different books. Find the number of ways of arranging all these books in a shelf in a single row.

4. A book store has 'm' copies each of 'n' different books. Find the number of ways of arranging these books in a shelf in a single row.

5. Find the number of 5-digit numbers that can be formed using the digits 0, 1, 1, 2, 3.

6. In how many ways can the letters of the word CHEESE be arranged so that no two E's come together?

III. 1. Find the number of ways of arranging the letters of the word ASSOCIATIONS. In how many of them

(i) all the three S's come together.

(ii) the two A's do not come together.

2. Find the number of ways of arranging the letters of the word MISSING so that the two S's are together and the two I's are together.

3. If the letters of the word AJANTA are permuted in all possible ways and the words thus formed are arranged in dictionary order, find the ranks of the words (i) AJANTA (ii) JANATA

5.6 Combinations - Definitions and Certain Theorems

At the beginning of this chapter, we have exhibited the difference between a permutation and a combination. A combination is only a selection. There is no importance to the order or arrangment of things in a combination. Thus a combination of 'n' things taken 'r' at a time can be regarded as a subset with 'r' elements of a set containing 'n' elements. The number of combinations of 'n' dissimilar things taken 'r' at a time is denoted by ${}^{n}C_{r}$ or $C(n, r)$ or $\binom{n}{r}$ and it is equal to the number of subsets with 'r' elements of set containing 'n' elements. In the succeeding theorem we develop a formula to find ${}^{n}C_{r}$.

5.6.1 Theorem

The number of combinations of 'n' dissimilar things taken 'r' at a time is

$$\frac{^nP_r}{r!}. \text{ That is } {}^nC_r = \frac{^nP_r}{r!} = \frac{n!}{(n-r)!r!}.$$

Proof : Let A be the set containing the given n dissimilar things. Then the number of combinations $({}^nC_r)$ of these n dissimilar things taken r at a time is equal to the number of subsets of A containing r-elements. If we select one such subset of A containing 'r' elements, these 'r' things can be arranged in a line in $r!$ ways. In other words, one combination gives rise to $r!$ permutations (of n things taken r at a time). Thus from these ${}^nC_r$ combinations we get ${}^nC_r \times r!$ permutations. But we know that the number of permutations of 'n' things taken 'r' at a time is ${}^nP_r$. Hence

$${}^nC_r \times r! = {}^nP_r$$

Therefore,
$${}^nC_r = \frac{^nP_r}{r!} = \frac{n!}{(n-r)!.r!}$$

5.6.2 Note : From the above theorem ${}^nC_r = \frac{^nP_r}{r!} = \frac{n(n-1)(n-2)...(n-r+1)}{r(r-1)(r-2)\ ...\ 1}$. Thus the numerator is the product of r consecutive integers in decreasing order starting from n while the denominator is the product of r consecutive integers in decreasing order, starting from r. For examples

$${}^9C_3 = \frac{9\times8\times7}{3\times2\times1} = 84 \text{ and } {}^{10}C_4 = \frac{10\times9\times8\times7}{1\times2\times3\times4} = 210.$$

The following is a direct consequence of Theorem 5.6.1.

5.6.3 Corollary : The number of different subsets of 'r' elements of a set containing 'n' elements is ${}^nC_r$.

5.6.4 Examples

1. Example : Find the number of ways of selecting 7 members from a contingent of 10 soldiers.

Solution : The number of ways of selecting 7 members out of 10 soldiers is

$${}^{10}C_7 = \frac{10!}{3!\,7!} = 120. \text{ (from theorem 5.6.1)}$$

2. Example : If a set A has 8 elements, find the number of subsets of A, containing at least 6 elements.

Solution : We have to fix the number of subsets of A, containing 6 or 7 or 8 elements.

Number of subsets of A, containing exactly 6 elements $= {}^8C_6$

Number of subsets of A, containing exactly 7 elements $= {}^8C_7$

Number of subsets of A, containing exactly 8 elements $= {}^8C_8$

Therefore, the number of subsets of A containing atleast 6 elements

$$= {}^8C_6 + {}^8C_7 + {}^8C_8 = 28 + 8 + 1 = 37.$$

Whenever we select r elements out of n elements, we will be left with $(n-r)$ elements. Thus, the number of ways of selecting 'r' elements from the given n elements is equal to the number of ways of leaving $(n-r)$ elements. This is proved in the following.

5.6.5 Theorem

If n, r are integers with $0 \le r \le n$, the ${}^nC_r = {}^nC_{(n-r)}$

Proof : ${}^nC_{(n-r)} = \dfrac{n!}{(n-(n-r))!.(n-r)!}$ (from theorem 5.6.1)

$$= \frac{n!}{r!.(n-r)!} = {}^nC_r.$$

5.6.6 Corollary : For any positive integer n, ${}^nC_n = {}^nC_0 = 1$.

Proof : From Theorem 5.6.1

$${}^nC_n = \frac{n!}{(n-n)!\,n!} = \frac{n!}{0!\,n!} = 1 \quad (\text{since } 0! = 1)$$

Taking $r = n$ in theorem 5.6.5, we get ${}^nC_n = {}^nC_{(n-n)} = {}^nC_0$.

Thus ${}^nC_n = {}^nC_0 = 1$.

5.6.7 Theorem

If m, n are distinct positive integers, then the number of ways of dividing $(m + n)$ things into two groups containing 'm' things and 'n' things is $\dfrac{(m+n)!}{m!\,n!}$.

Proof : Whenever we select 'm' things out of the given $(m + n)$ things, we will be left with n things and hence two groups one containing m things and the other containing 'n' things are formed. Therefore, the number of ways of dividing $(m+n)$ dissimilar things into two groups containing 'm' things out of 'n' things.

= The number of ways of selecting 'm' things out of $(m+n)$ things

$$= {}^{(m+n)}C_m = \frac{(m+n)!}{(m+n-m)!.m!} = \frac{(m+n)!}{m!.n!}$$

5.6.8 Corollary : If m, n, p are distinct positive integers, then the number of ways of dividing '$m+n+p$' things into 3 groups containing 'm' things, 'n' things and 'p' things is

$$\frac{(m+n+p)!}{m!.n!.p!}$$

Proof : First we select 'm' things from the given '$m+n+p$' things to form the first group in ${}^{(m+n+p)}C_m$ ways. Now, from the remaining $(n+p)$ things we select 'n' things to form second group in ${}^{(n+p)}C_n$ ways. Then the remaining 'p' things automatically form the third group. Thus the number of ways of dividing the given $(m+n+p)$ dissimilar things into 3 groups containing m, n, p things

$$= {}^{(m+n+p)}C_m \times {}^{(n+p)}C_n = \frac{(m+n+p)!}{(m+n+p-m)!\,m!} \times \frac{(n+p)!}{(n+p-n)!\,n!}$$

$$= \frac{(m+n+p)!}{(n+p)!\,m!} \times \frac{(n+p)!}{p!\,n!} = \frac{(m+n+p)!}{m!\,n!\,p!}.$$

5.6.9 Corollary : The number of ways of dividing $2n$ dissimilar things into two equal groups containing 'n' things in each is $\frac{(2n)!}{2!\,n!\,n!}$.

Proof : As in Theorem 5.6.7, we can divide '$2n$' things into two groups of n elements each in $\frac{2n!}{n!\,n!}$ ways. Since the two groups have equal number of elements, we can interchange them in 2! ways. Though they have given rise to the same division, they are counted as '2' divisions in the above calculation. Hence we can divide the given '$2n$' things into two equal groups having n elements in each group in

$$\frac{2n!}{2!\,n!\,n!} \text{ ways}$$

We now extend this result to mn things in the following corollary.

5.6.10 Corollary : The number of ways of dividing 'mn' dissimilar things into 'm' equal groups each containing 'n' elements is

$$\frac{(mn)!}{m!(n!)^m}$$

If we have to distribute 'mn' things equally among 'm' persons we use the following.

5.6.11 Corollary : The number of ways of distributing 'mn' dissimilar things equally among 'm' persons is

$$\frac{(mn)!}{(n!)^m}$$

Proof : First we divide 'mn' things into 'm' equal groups (using corollary 5.6.10) in $\frac{(mn)!}{m!(n!)^m}$ ways. Now we have m groups (not identical, they contain only equal number of elements) and m persons. Hence they can be distributed in $m!$ ways. Hence, by the fundamental principle, the number of ways of distributing 'mn' things to 'm' persons equally is

$$\frac{(mn)!}{m!(n!)^m} \times m! = \frac{(mn)!}{(n!)^m}$$

5.6.12 Example : A candidate is required to answer 6 out of 10 questions which are divided into two groups A and B each containing 5 questions. He is not permitted to attempt more than 4 questions from either group. Find the number of different ways in which the candidate can choose six questions.

Solution : The candidate can answer 4 questions from group A and 2 questions from group B or 3 questions from each group or 2 questions from group A and 4 questions from group B. The number of ways of choosing questions by the candidate with

(i) 4 from group A and 2 from group B $= {}^5C_4 \times {}^5C_2 = 50.$

(ii) 3 from each group $= {}^5C_3 \times {}^5C_3 = 100.$

(iii) 2 from group A and 4 from group B $= {}^5C_2 \times {}^5C_4 = 50.$

Thus the number of ways of selecting 6 questions out of 10 questions is $50 + 100 + 50 = 200.$

5.6.13 Solved Problems

1. Problem : *Find the number of ways of selecting 4 boys and 3 girls from a group of 8 boys and 5 girls.*

Solution : 4 boys can be selected from the given 8 boys in ${}^8C_4$ ways and 3 girls can be selected from the given 5 girls in ${}^5C_3$ ways. Hence, by the Fundamental principle, the number of required selections is

$${}^8C_4 \times {}^5C_3 = 70 \times 10 = 700.$$

2. Problem : *Find the number of ways of selecting 4 English, 3 Telugu and 2 Hindi books out of 7 English, 6 Telugu and 5 Hindi books.*

Solution : The number of ways of selecting

4 English books out of 7 books $= {}^7C_4$

3 Telugu books out of 6 books $= {}^6C_3$

2 Hindi books out of 5 books $= {}^5C_2$

Hence, the number of required ways $= {}^7C_4 \times {}^6C_3 \times {}^5C_2 = 35 \times 20 \times 10 = 7000.$

3. Problem : *Find the number of ways of forming a committee of 4 members out of 6 boys and 4 girls such that there is atleast one girl in the committee.*

Solution : The number of ways of forming a committee of 4 members out of 10 members (6 boys + 4 girls) is ${}^{10}C_4$. Out of these, the number of ways of forming the committee having no girl is ${}^{6}C_4$ (we select all 4 members from boys). Therefore, the number of ways of forming the committees having atleast one girl is

$$ {}^{10}C_4 - {}^{6}C_4 = 210 - 15 = 195. $$

4. Problem : *Find the number of ways of selecting 11 member cricket team from 7 bats men, 6 bowlers and 2 wicket keepers so that the team contains 2 wicket keepers and atleast 4 bowlers.*

Solution : The required cricket team can have the following compositions

Bowlers	Wicket Keepers	Batsmen	Number of ways of selecting team
4	2	5	${}^{6}C_4 \times {}^{2}C_2 \times {}^{7}C_5 = 15 \times 1 \times 21 = 315$
5	2	4	${}^{6}C_5 \times {}^{2}C_2 \times {}^{7}C_4 = 6 \times 1 \times 35 = 210$
6	2	3	${}^{6}C_6 \times {}^{2}C_2 \times {}^{7}C_3 = 1 \times 1 \times 35 = 35$

Therefore, the number of ways of selecting the required cricket team

$$ = 315 + 210 + 35 = 560. $$

5. Problem : *If a set of 'm' parallel lines intersect another set of 'n' parallel lines (not parallel to the lines in the first set), then find the number of parallelograms formed in this lattice structure.*

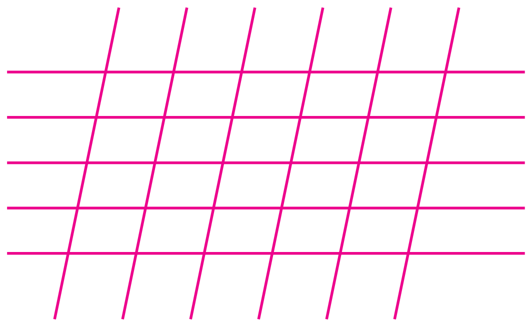


Fig. 5.6

Solution : Whenever we select 2 lines from the first set of m lines and 2 lines from the second set of n lines, one parallelogram is formed as shown in the figure.

Thus, the number of parallelograms formed is ${}^{m}C_2 \times {}^{n}C_2$.

6. Problem : *There are 'm' points in a plane out of which 'p' points are collinear and no three of the points are collinear unless all the three are from these p points. Find the number of different*

(i) *straight lines passing through pairs of distinct points.*

(ii) *triangles formed by joining these points (by line segments).*

Solution

(i) From the given 'm' points, by drawing straight lines passing through 2 distinct points at a time, we are supposed to get '${}^{m}C_2$' number of lines. But, since 'p' out of these 'm' points are collinear, by forming lines passing through these 'p' points 2 at a time we get only one line instead of getting ${}^{p}C_2$. Therefore, the number of different lines as required is

$$ {}^{m}C_2 - {}^{p}C_2 + 1. $$

(ii) From the given m points, by joining 3 at a time, we are supposed to get ${}^{m}C_3$ number of triangles. Since p points out of these m points are collinear, by joining these p points 3 at a time we do not get any triangle (we get only a line) when we are supposed to get ${}^{p}C_3$ number of triangles. Hence the number of triangles formed by joining the given m points is

$$ {}^{m}C_3 - {}^{p}C_3. $$

Note : The number of diagonals in an n-sided polygon $= {}^{n}C_2 - n = \dfrac{n(n-3)}{2}$.

7. Problem : *A teacher wants to take 10 students to a park. He can take exactly 3 students at a time and will not take the same group of 3 students more than once. Find the number of times*

(i) each student can go to the park (ii) the teacher can go to the park

Solution

(i) To find the number of times a student can go to the park, we have to select 2 more students from the remaining 9 students. This can be done in ${}^{9}C_2$ ways. Hence, each student can go to park ${}^{9}C_2 = 36$ times.

(ii) The number of times the teacher can go to park

= The number of different ways of selecting 3 students out of 10.

$= {}^{10}C_3 = 120.$

8. Problem : *A double decker minibus has 8 seats in the lower deck and 10 seats in the upper deck. Find the number of ways of arranging 18 persons in the bus if 3 children want to go to the upper deck and 4 old people can not go to the upper deck.*

Solution

Allowing 3 children to the upper deck and 4 old people to the lower deck, we are left with 11 people and 11 seats (7 in the upper deck and 4 in the lower deck). We can select 7 people for the upper deck out of the 11 people in ${}^{11}C_7$ ways. The remaining 4 persons go to lower deck. Now we can arrange 10 persons (3 children and 7 others) in the upper deck and 8 persons (4 old people and 4 others) in the lower deck in 10! and 8! ways respectively. Hence, the required number of arrangements

$= {}^{11}C_7 \times 10!\ 8!.$

5.6.14 Certain theorems on combinations

In this section, we prove two important theorems about ${}^nC_r$. In theorem 5.6.5, we have proved that ${}^nC_r = {}^nC_{n-r}$. In this section, we prove the converse of this result. Before that, we prove the following lemma which will be used to prove the theorem.

5.6.15 Lemma : If a, b are positive real numbers and k is a positive integer such that

$$(a+1)(a+2)\ldots(a+k) = (b+1)(b+2)\ldots(b+k),$$

then $a = b$

Proof : Suppose $a \neq b$, Then either $a < b$ or $a > b$. Without loss of generality assume $a < b$. Then, for $1 \le i \le k$, we have $b + i > a + i$.

on multiplying these inequalities, we get

$$(b+1)(b+2)\ldots(b+k) > (a+1)(a+2)\ldots(a+k)$$

which is a contradiction to the hypothesis. Hence $a = b$.

Now, we prove the following.

5.6.16 Theorem : *For $0 \le r, s \le n$, if ${}^nC_r = {}^nC_s$, then either $r = s$ or $r + s = n$. (that is, either $s = r$ or $s = n - r$)*

Proof : Suppose ${}^nC_r = {}^nC_s$. If $r = s$, the theorem is proved. Assume $r \neq s$. Without loss of generality assume $r < s$. Then $(n-s) < (n-r)$. Now

$${}^nC_r = {}^nC_s \Rightarrow \frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-s)!\,s!}$$

$$\Rightarrow (n-r)!\, r! = (n-s)!\, s!$$

$$\Rightarrow [(n-r)(n-r-1)\ldots.(n-s+1)](n-s)!\, r! = (n-s)!\,[s(s-1)\ldots(r+1)]\,r!$$

(since $r < s$ and $(n-s) < (n-r)$)

$$\Rightarrow (n-r)(n-r-1)\ldots(n-s+1) = s(s-1)\ldots(r+1)$$

Hence by taking $a = n - s + 1$, $b = r + 1$ and $k = s - r$ in Lemma 5.6.15, we get

$$n - s + 1 = r + 1. \quad \text{Hence } r = n - s \text{ or } n = r + s.$$

Now, we prove another important property of ${}^nC_r$ which is analogous to Theorem 5.2.10

5.6.17 Theorem : *If $1 \leq r \leq n$, then* $\boxed{{}^nC_{r-1} + {}^nC_r = {}^{(n+1)}C_r.}$

Proof

First method : We know that ${}^{(n+1)}C_r$ is the number of r–element subsets (That is, subsets having r–elements) of a set A containing $(n+1)$ elements. Fix $a \in$ A. If we select an r–element subset of A, it may or may not contain the fixed element a. We calculate the number of such subsets now.

(i) Number of r–element subsets of A containing the element $'a'$

= Number of ways of selecting $(r-1)$ elements from the remaining n elements of A(since already one element, that is, a is selected)

$= {}^nC_{(r-1)}$

(ii) Number of r–element subsets of A not containing the element $'a'$

= Number of ways of selecting r–elements from the remaining n elements of A (leaving the element a)

$= {}^nC_r$

Thus, from (i) and (ii), we get

$$ {}^{(n+1)}C_r = {}^nC_{(r-1)} + {}^nC_r $$

Second Method : (Using the formula for ${}^nC_r$)

$$ {}^nC_{(r-1)} + {}^nC_r = \frac{n!}{(n-(r-1))!(r-1)!} + \frac{n!}{(n-r)!r!} $$

$$ = n!\left\{\frac{1}{(n-r+1)!(r-1)!} + \frac{1}{(n-r)!r!}\right\} $$

$$ = n!\left\{\frac{r+(n-r+1)}{(n-r+1)!r!}\right\} = \frac{n!(n+1)}{(n-r+1)!r!} $$

$$ = \frac{(n+1)!}{((n+1)-r)!r!} = {}^{(n+1)}C_r $$

5.6.18 Corollary : If $2 \leq r \leq n$, then

$$ {}^nC_{(r-2)} + 2.{}^nC_{(r-1)} + {}^nC_r = {}^{(n+2)}C_r $$

Proof : L.H.S. $= {}^nC_{(r-2)} + 2.{}^nC_{(r-1)} + {}^nC_r$

$= ({}^nC_{(r-2)} + {}^nC_{(r-1)}) + ({}^nC_{(r-1)} + {}^nC_r)$

$$= {}^{(n+1)}C_{(r-1)} + {}^{(n+1)}C_r \text{ (by Theorem 5.6.17)}$$

$$= {}^{(n+2)}C_r \text{ (again by Theorem 5.6.17)}$$

$$= \text{R.H.S.}$$

5.6.19 Theorem : *It p things are alike of one kind, q things are alike of second kind and r things are alike of third kind, then the number of ways of selecting any number of things (one or more) out of these (p + q + r) things is (p + 1) (q + 1) (r + 1) − 1.*

Proof : From the first group of p things, we can select 0 or 1 or 2 or ... or p things. Since all the p things in this group are alike, we have to decide only the number of things to be selected. This can be done in $(p+1)$ ways. Similarly, we can select any number of things from second and third groups (up to a maximum of q and r things respectively) in $(q+1)$ and $(r+1)$ ways respectively. Hence, by the fundamental principle, we can select any number of things from the 3 groups in

$$(p+1) \,.\, (q+1) \,.\, (r+1)$$

ways. But this includes one selecting of '0' from each group. Since we have to select one or more things, the number of required ways is

$$(p+1)\,(q+1)\,(r+1) - 1.$$

5.6.20 Corollary : The number of ways of selecting one or more things out of 'n' dissimilar things is $2^n - 1$.

Proof : The given n dissimilar things can be regarded as n groups having 1 alike thing in each group. Hence, by theorem 5.6.19, the number of ways of selecting one or more things out of the given n things is

$$\underbrace{(1+1)\,(1+1)\,...\,(1+1)}_{(n \text{ times})} - 1 = 2^n - 1$$

5.6.21 Corollary : If $p_1, p_2, ..., p_k$ are distinct primes and $\alpha_1, \alpha_2, ..., \alpha_k$ are positive integers, then the number of positive divisors of

$$n = p_1^{\alpha_1} \,.\, p_2^{\alpha_2} \,...\, p_k^{\alpha_k} \text{ is } (\alpha_1+1)\,(\alpha_2+1)\,...\,(\alpha_k+1) \text{ (this includes 1 and } n).$$

Proof : If $n = p_1^{\alpha_1} \,.\, p_2^{\alpha_2} \,...\, p_k^{\alpha_k}$, then any positive divisor of n is of the form $p_1^{\beta_1} \,.\, p_2^{\beta_2} \,...\, p_k^{\beta_k}$ where $0 \le \beta_i \le \alpha_i$ for $1 \le i \le k$. Thus β_i can be 0 or 1 or 2 or ... or α_i, that is β_i can take $\alpha_i + 1$ values for $1 \le i \le k$. Hence, by Theorem 5.6.19, the number of positive divisors of n is $(\alpha_1 + 1)\,(\alpha_2 + 1)\,...\,(\alpha_k + 1)$. (Since when all β_i's are zeroes we get the divisor 1).

5.6.22 Note

The positive divisors of n, other than 1 and n itself are called proper divisors of n and 1, n are called improper (or trivial) divisors of n. Thus, the number of proper divisors of a positive integer $n = p_1^{\alpha_1} . p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, (where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are non-negative integers) is $(\alpha_1 + 1)(\alpha_2 + 1) \ldots (\alpha_k + 1) - 2$.

5.6.23 Definition

Exponent of a prime in n! ($n \in Z^+$) : *Exponent of a prime p in n! is the largest integer k such that p^k divides n!.*

The exponent of p in $n!$ is given by

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots.$$

(proof of this result is beyond the scope of this book)

5.6.24 Example : Find the number of zeros in 100!.

Solution : $100! = 2^{\alpha}\, 3^{\beta}\, 5^{\gamma}\, 7^{\delta} \ldots.$

$$\text{where } \alpha = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \ldots.$$

$$= 50 + 25 + 12 + 6 + 3 + 1$$

$$= 97$$

$$\gamma = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right]$$

$$= 20 + 4 = 24.$$

Now, the number of zeros in 100! is the power of 10 in 100! which is 24, since $10 = 2 \times 5$.

5.6.25 Example : If there are 5 alike pens, 6 alike pencils and 7 alike erasers, find the number of ways of selecting any number of (one or more) things out of them.

Solution : By Theorem 5.6.19, the required number of ways is

$$(5 + 1)(6 + 1)(7 + 1) - 1 = 335.$$

5.6.26 Example : Find the number of positive divisors of 1080.

Solution : $1080 = 2^3 \times 3^3 \times 5^1$

Hence, by corollary 5.6.21, the number of postive divisors of $1080 = (3+1)(3+1)(1+1) = 32$.

5.6.27 Example : To pass an examination a student has to pass in each of the three papers. In how many ways can a student fail in the examination?

Solution : For each of the three papers there are two choices P or F. There are $2^3 = 8$ choices. But a student passes only if he/she passes in all papers.

$\therefore$ Required no. of ways $= 2^3 - 1 = 7$.

5.6.28 Example : Out of 3 different books on Economics, 4 different books on political science and 5 different books on Geography, how many collections can be made, if each collection consists of (i) exactly one book of each subject (ii) atleast one book of each subject.

Solution

(i) Out of 3 books on Economics exactly one book is chosen in ${}^3C_1$ ways.

Out of 4 books on political science one book can be chosen in ${}^4C_1$ ways, and a Geography book out of 5 books on it can be chosen in ${}^5C_1$ ways.

$\therefore$ Required number of ways $= {}^3C_1 \, . \, {}^4C_1 \, . \, {}^5C_1 = 3 \times 4 \times 5 = 60$.

(ii) As in Example 5.6.27, the number of collections having atleast one book of each subject is

$$(2^3 - 1)(2^4 - 1)(2^5 - 1) = 7 \times 15 \times 31 = 3255.$$

Note: In the above problem, if the books of each subject are alike then the required number of ways are (i) 1, (ii) $3 \times 4 \times 5 = 60$ respectively.

5.6.29 Solved Problems

1. Problem : *Prove that*

(i) ${}^{10}C_3 + {}^{10}C_6 = {}^{11}C_4$

(ii) ${}^{25}C_4 + \sum_{r=0}^{4} {}^{(29-r)}C_3 = {}^{30}C_4$

Solution

(i) ${}^{10}C_3 + {}^{10}C_6 = {}^{10}C_3 + {}^{10}C_4$ (since ${}^nC_r = {}^nC_{(n-r)}$)

$= {}^{11}C_4$ (by Theorem 5.6.17)

(ii) $^{25}C_4 + \sum_{r=0}^{4} {}^{(29-r)}C_3$

$= {}^{25}C_4 + \{{}^{25}C_3 + {}^{26}C_3 + {}^{27}C_3 + {}^{28}C_3 + {}^{29}C_3\}$

$= {}^{26}C_4 + {}^{26}C_3 + {}^{27}C_3 + {}^{28}C_3 + {}^{29}C_3$ (since $^{25}C_3 + {}^{25}C_4 = {}^{26}C_4$)

$= {}^{27}C_4 + {}^{27}C_3 + {}^{28}C_3 + {}^{29}C_3$

$= {}^{28}C_4 + {}^{28}C_3 + {}^{29}C_3$

$= {}^{29}C_4 + {}^{29}C_3$

$= {}^{30}C_4.$

2. Problem : (i) *If* $^{12}C_{(s+1)} = {}^{12}C_{(2s-5)}$, *find* s

(ii) *If* $^{n}C_{21} = {}^{n}C_{27}$, *find* $^{50}C_n$.

Solution

(i) By Theorem 5.6.16,

$$^{12}C_{(s+1)} = {}^{12}C_{(2s-5)} \Rightarrow \text{either } s+1 = 2s-5 \text{ or } (s+1)+(2s-5) = 12$$

$$\Rightarrow s = 6 \text{ or } s = \frac{16}{3}.$$

$$\Rightarrow s = 6 \quad \text{(since } s \text{ is a non negative integer)}$$

(ii) By Theorem 5.6.16,

$$^{n}C_{21} = {}^{n}C_{27} \Rightarrow n = 21 + 27 = 48$$

Therefore, $^{50}C_n = {}^{50}C_{48} = {}^{50}C_2 = \dfrac{50\times 49}{1\times 2} = 1225.$

3. Problem : *14 persons are seated at a round table. Find the number of ways of selecting two persons out of them who are not seated adjacent to each other.*

Solution : Let the seating arrangement of given 14 persons at the round table be as shown in Fig. 5.7.

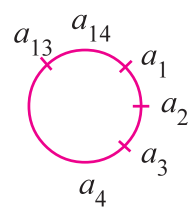


Fig. 5.7

Number of ways of selecting 2 persons out of 14 persons $^{14}C_2 = 91$.

In the above arrangement two persons sitting adjacent to each other can be selected in 14 ways (they are $a_1 a_2, a_2 a_3, \ldots, a_{13} a_{14}, a_{14} a_1$).

Therefore, the required number of ways $= 91 - 14 = 77$.

Exercise 5(e)

I. 1. If ${}^{n}C_4 = 210$, find n.

2. If ${}^{12}C_r = 495$, find the possible values of 'r'.

3. If $10.{}^{n}C_2 = 3.{}^{n+1}C_3$, find n.

4. If ${}^{n}P_r = 5040$ and ${}^{n}C_r = 210$, find n and r.

5. If ${}^{n}C_4 = {}^{n}C_6$, find n.

6. If ${}^{15}C_{2r-1} = {}^{15}C_{2r+4}$, find r.

7. If ${}^{17}C_{2t+1} = {}^{17}C_{3t-5}$, find t.

8. If ${}^{12}C_{r+1} = {}^{12}C_{3r-5}$, find r.

9. If ${}^{9}C_3 + {}^{9}C_5 = {}^{10}C_r$, then find r.

10. Find the number of ways of forming a committee of 5 members from 6 men and 3 ladies.

11. In question no. 10, how many committees contain atleast two ladies.

12. If ${}^{n}C_5 = {}^{n}C_6$, then find ${}^{13}C_n$.

II. 1. Prove that for $3 \le r \le n$,

$$^{(n-3)}C_r + 3.{}^{(n-3)}C_{(r-1)} + 3.{}^{(n-3)}C_{(r-2)} + {}^{(n-3)}C_{(r-3)} = {}^{n}C_r.$$

2. Find the value of ${}^{10}C_5 + 2.{}^{10}C_4 + {}^{10}C_3$.

3. Simplify ${}^{34}C_5 + \sum_{r=0}^{4} {}^{(38-r)}C_4$.

4. In a class there are 30 students. If each student plays a chess game with each of the other student, then find the total number of chess games played by them.

5. Find the number of ways of selecting 3 girls and 3 boys out of 7 girls and 6 boys.

6. Find the number of ways of selecting a committee of 6 members out of 10 members always including a specified member.

7. Find the number of ways of selecting 5 books from 9 different mathematics books such that a particular book is not included.

8. Find the number of ways of selecting 3 vowels and 2 consonants from the letters of the word EQUATION.

9. Find the number of diagonals of a polygon with 12 sides.

10. If n persons are sitting in a row, find the number of ways of selecting two persons, who are sitting adjacent to each other.

11. Find the number of ways of giving away 4 similar coins to 5 boys if each boy can be given any number (less than or equal to 4) of coins.

III. 1. Prove that $\dfrac{{}^{4n}C_{2n}}{{}^{2n}C_{n}} = \dfrac{1.3.5 \ldots (4n-1)}{\{1.3.5 \ldots (2n-1)\}^2}$

2. If a set A has 12 elements, find the number of subsets of A having

   (i) 4 elements (ii) Atleast 3 elements (iii) Atmost 3 elements.

3. Find the numbers of ways of selecting a cricket team of 11 players from 7 batsmen and 6 bowlers such that there will be atleast 5 bowlers in the team.

4. If 5 vowels and 6 consonants are given, then how many 6 letter words can be formed with 3 vowels and 3 consonants.

5. There are 8 railway stations along a railway line. In how many ways can a train be stopped at 3 of these stations such that no two of them are consecutive?

6. Find the number of ways of forming a committee of 5 members out of 6 Indians and 5 Americans so that always the Indians will be in majority in the committee.

7. A question paper is divided into 3 sections A, B, C containing 3, 4, 5 questions respectively. Find the number of ways of attempting 6 questions choosing atleast one from each section.

8. Find the number of ways in which 12 things be (i) divided into 4 equal groups (ii) distributed to 4 persons equally.

9. A class contains 4 boys and g girls. Every sunday, five students with atleast 3 boys go for a picnic. A different group is being sent every week. During the picnic, the class teacher gives each girl in the group a doll. If the total number of dolls distributed is 85, find g.

Key Concepts

Here we give a brief summary of the results and important concepts of this chapter.

❖ Fundamental principle : If a work W_1 can be performed in m different ways and another work W_2 can be performed in n different ways, then the two works simultaneously can be performed in mn different ways.

❖ If n is a positive integer, then $n! = n\{(n-1)!\}$ and $1! = 1$.

❖ We define $0! = 1$.

❖ The number of permutations of n dissimilar things taken 'r' at a time is denoted by ${}^nP_r$ and ${}^nP_r = \dfrac{n!}{(n-r)!}$ for $0 \le r \le n$.

❖ If n, r are positive integers and $r \le n$, then

(i) ${}^nP_r = n \,.\, {}^{(n-1)}P_{(r-1)}$ (if $r \ge 1$) (ii) ${}^nP_r = n \,.\, (n-1) \,.\, {}^{(n-2)}P_{(r-2)}$. (If $r \ge 2$)

❖ The number of permutations of n dissimilar things taken 'r' at a time

(i) containing a particular thing is $r \,.\, {}^{(n-1)}P_{(r-1)}$.

(ii) not containing a particular thing is ${}^{(n-1)}P_r$.

(iii) containing a particular thing in a particular place is ${}^{(n-1)}P_{(r-1)}$.

❖ If n, r are positive integers and $r \le n$, then ${}^nP_r = {}^{(n-1)}P_r + r \,.\, {}^{(n-1)}P_{(r-1)}$.

❖ The sum of the r-digit numbers that can be formed using the given 'n' distinct non-zero digits ($r \le n \le 9$) is ${}^{(n-1)}P_{(r-1)} \times$ (sum of all n digits) $\times$ $(111 \,....\, 1)_{(r\text{ times})}$).

❖ In the above, if '0' is one among the given 'n' digits, then the sum is ${}^{(n-1)}P_{(r-1)} \times$ (sum of the digits) $\times$ $111... 1_{(r\text{ times})} - ({}^{(n-2)}P_{(r-2)} \times$ (sum of the digits) $\times$ $(111 \,...\, 1)_{(r-1)\text{ times}}$).

❖ The number of permutations of n dissimilar things taken 'r' at a time when repetitions are allowed [i.e., each thing can be used any number of times) is n^r.

❖ The number of circular permutations of n dissimilar things is $(n-1)!$.

❖ In the case of hanging type circular permutations like garlands of flowers, chains of beads etc., the number of circular permutations of n things is $\frac{1}{2}\{(n-1)!\}$.

❖ If in the given n things, p like things are of one kind, q alike things are of the second kind, r alike things are of the third kind and the rest are dissimilar, then the number of permutations (of these n things) is $\frac{n!}{(p!)(q!)(r!)}$.

❖ The number of combinations of n things taken 'r' at a time is denoted by ${}^nC_r$ and ${}^nC_r = \frac{n!}{(n-r)!r!}$ for $0 \le r \le n$.

❖ If n, r are integers and $0 \le r \le n$, then ${}^nC_r = {}^nC_{(n-r)}$.

❖ ${}^nC_0 = {}^nC_n$; ${}^nC_1 = {}^nC_{(n-1)}$.

❖ The number of ways of dividing '$m + n$' things ($m \neq n$) into two groups containing m, n things is ${}^{(m+n)}C_m = {}^{(m+n)}C_n = \frac{(m+n)!}{m!n!}$.

❖ The number of ways of dividing ($m + n + p$) things (m, n, p are distinct) into 3 groups of m, n, p things is $\frac{(m+n+p)!}{(m!)(n!)(p!)}$.

❖ The number of ways of dividing mn things into m equal groups is $\frac{(mn)!}{(n!)^m\,(m!)}$.

❖ The number of ways if distributing mn things equally to m persons is $\frac{(mn)!}{(n!)^m}$.

❖ If p alike things are of one kind, q alike things are of the second kind, and r alike things are of the third kind, then the number of ways of selecting one or more things out of them is $(p+1)(q+1)(r+1) - 1$.

❖ If m is a positive integer and

$$m = p_1^{\alpha_1} . p_2^{\alpha_2} \dots p_k^{\alpha_k}$$

where $p_1, p_2, \dots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \dots, \alpha_k$ are positive integers, then the number of divisors of m is $(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_k + 1)$ (This includes 1 and m).

Historical Note

Ancient Indian Pandit *Pingala's Chandassastra* (ca–200 B.C.) contains material on the theory of permutations and combinations. *Mahaviracharya* and *Bhaskaracharya* (9th Century A.D., 12th Century A.D.) contributed substantially to this topic.

A Hebrew writer *Rabbi Ben Ezra* determined the number of combinations of the then known planets by taking one at a time, two at a time etc., as 1140 approximately. Though he did not know the formula for ${}^nC_r$, he observed that the number of combinations of n planets, taken r at a time is the same as the number of combinations taken $(n-r)$ at a time for certain values of $r\ (\le n)$. The formulae for ${}^nP_r$, ${}^nC_r$ have been given by another Hebrew writer '*Levi Ben Gerson*'.

Answers

Exercise 5(a)

I. 1. 12   2. 12   3. 11   4. 5   5. 5

6. ${}^5P_4 = 120$

II. 1. ${}^{25}P_2 = 600$ 2. ${}^{30}P_2 = 870$ 3. $5! \times 3! = 720$

4. ${}^4P_3 \times 21 \times 1111 - {}^3P_2 \times 21 \times 111 = 5{,}45{,}958$ 5. $3 \times {}^4P_3 + 4 \times 4! = 168$

6. ${}^3P_2 \times 4! = 144$ 7. $3! \times 5! \times 4! \times 3! = 1{,}03{,}680$

III. 1. ${}^8P_5 = 6720$; ${}^7P_4 = 840$; ${}^7P_4 = 840$; ${}^6P_3 = 120$

2. (i) $(8!)\,(3!)$ (ii) $\dfrac{10!}{3!}$ (iii) $8!$

3. (i) $4!\ 5!$ (ii) $2!\ 5!\ 4!$

4. ${}^5P_4 = 120$

(i) $2! \times {}^4P_3 = 48$ (ii) $4! = 24$ (iii) $6 \times {}^3P_2 = 36$

(iv) ${}^4P_3 = 24$ (v) $2 \times {}^3P_2 = 12$

5. (i) 391 (ii) 257

6. IGRBN 7. ${}^4P_3 \times 18 \times 1111 = 4, 79, 952$

8. ${}^6P_5\ 4! = 17280$

Exercise 5(b)

I. 1. $6^4 = 1296$ 2. $5^5 = 3125$ 3. $4^5 = 1024$

II. 1 (i) 4×5^2 (ii) 5^3 2. $6^4 - {}^6P_4 = 936$

3. ${}^7P_7 = 5040$ 4. $n^r - {}^nP_r$ 5. $6^4 = 1296$

6. $2 \times 5 \times 6^3 = 2160$ 7. $4 + 4^2 + 4^3 + 4^3.1 = 148.$

III. 1. (i) ${}^9P_4 = 3024$ (ii) $9^4 - {}^9P_4 = 3537$

2. (i) $3 \times 4 \times 5^2 = 300$ (ii) 160 3. $5 \times 6^2 \times 1 = 180$

Exercise 5(c)

I. 1. $(7-1)! = 720$ 2. $(10)!$ 3. $\frac{1}{2}[(6-1)!] = 60$

II. 1. $4! \times 3! = 144$ 2. $6! \times {}^7P_4$ 3. $5! \times 2! = 240$

4. $\frac{1}{2}(7! \times 2!) = 5040$

III. 1. (i) $6! \times 6!$ (ii) $5! \times 6!$ (iii) $5! \times 6!$

2. $\frac{1}{2}[(9-1)!] = 20,160$ (i) $\frac{6! \times 3!}{2} = 2160$ (ii) $\frac{5! \times {}^6P_3}{2} = 7200$

3. $3! \times 3! \times 3! \times 3! \times 2! = 2,592$ 4. $\frac{1}{2}(5! \times {}^6P_3) = 7,200$

5. $3! \times 2! = 12$

Exercise 5(d)

I. 1. (i) $\frac{(12)!}{4!.3!.2!}$ (ii) $\frac{(11)!}{2!.2!.2!}$ (iii) $\frac{7!}{2!.2!.2!}$

(iv) $\frac{(11)!}{2!}$ (v) $\frac{(11)!}{2!.2!.2!}$ (vi) $\frac{(12)!}{3!.2!.2!}$

2. $\frac{7!}{3!.2!.2!}$

II. 1. $4!+3\times\frac{4!}{2!}+3\times\frac{4!}{3!} = 72$ 2. $\frac{3!}{2!}\times\frac{4!}{2!\times2!} = 18$

3. $\frac{(35)!}{(3!)^2(4!)^2(5!)^3.6!}$ 4. $\frac{(mn)!}{(m!)^n}$ 5. $4!+2\cdot\frac{4!}{2!} = 48$

6. $3!\times\frac{{}^4P_3}{3!} = 24$

III. 1. $\frac{(12)!}{3!\ 2!\ 2!\ 2!}$ (i) $\frac{(10)!}{2!\ 2!\ 2!}$ (ii) $\frac{(10)!}{3!\ 2!\ 2!}\times\frac{{}^{11}P_2}{2!}$

2. $5! = 120$ 3. (i) 28 (ii) 68

Exercise 5(e)

I. 1. 10 2. 4 or 8 3. 9

4. $n = 10,\ r = 4$ 5. 10 6. 3

6. 6 8. 3 or 4 9. 4 or 6

10. ${}^9C_5 = 126$

11. ${}^3C_2 \times {}^6C_3 + {}^3C_3 \times {}^6C_2 = 75$

12. ${}^{13}C_{11} = 78$

II. 2. ${}^{12}C_5 = 792$ 3. ${}^{39}C_5$ 4. ${}^{30}C_2 = 435$

5. ${}^7C_3 \times {}^6C_3 = 700$ 6. ${}^9C_5 = 126$ 7. ${}^8C_5 = 56$

8. ${}^{5}C_{3} \times {}^{3}C_{2} = 30$ 9. ${}^{12}C_{2} - 12$ 10. $n - 1$

11. ${}^{5}C_{1} + {}^{5}C_{2} \times {}^{2}C_{1} + {}^{5}C_{2} + {}^{5}C_{3} \times {}^{3}C_{1} + {}^{5}C_{4} = 70$

III. 2. (i) ${}^{12}C_{4} = 495$

(ii) $2^{12} - \{{}^{12}C_{0} + {}^{12}C_{1} + {}^{12}C_{2}\} = 4017$

(iii) ${}^{12}C_{0} + {}^{12}C_{1} + {}^{12}C_{2} + {}^{12}C_{3} = 299$

3. ${}^{6}C_{5} \times {}^{7}C_{6} + {}^{6}C_{6} \times {}^{7}C_{5} = 63$

4. ${}^{5}C_{3} \times {}^{6}C_{3} \times 6!$

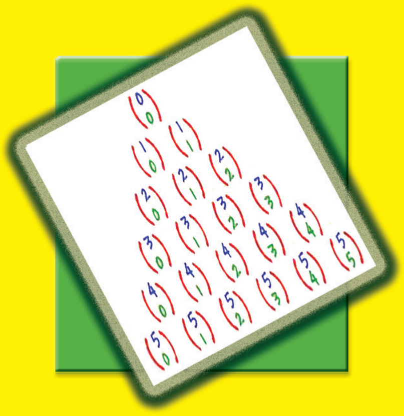
5. ${}^{8}C_{3} - 6 - 30 = 20$

6. ${}^{6}C_{3} \times {}^{5}C_{2} + {}^{6}C_{4} \times {}^{5}C_{1} + {}^{6}C_{5} = 281$

7. ${}^{12}C_{6} - {}^{7}C_{6} - {}^{9}C_{6} - {}^{8}C_{6} = 805$

8. (i) $\dfrac{(12)!}{(3!)^{4}4!}$ (ii) $\dfrac{(12)!}{(3!)^{4}}$

9. g = 5.



Chapter 6

Binomial Theorem

" It is much better to do a little with certainity, and leave the rest for others that come after you than to explain all things"

- Isaac Newton

Blaise Pascal

(1623 - 1662)

Pascal, *a French mathematician is famous for his work on projective geometry, arithmetical triangle, computing machines and for his unusual talents in physics and mechanical devices. The arithmetical triangle, also called pascal triangle is related to the coefficients of Binomial expansion.*

Introduction

Binomial means two terms connected by either '+' or '–'. We have come across many expansions of Squares, Cubes etc. of a binomial in earlier classes. For example,

$$(x+y)^1 = x^1 + y^1 = x + y$$

$$(x-y)^1 = x^1 - y^1 = x - y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x-y)^2 = x^2 - 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Each of these is an expansion of a power of the sum or difference of two terms. These are called **binomial expansions.** The coefficients 1, 1 in the expansion of $(x+y)^1$, 1, 2, 1 in the expansion of $(x+y)^2$, 1, 3, 3, 1 in $(x+y)^3$, 1, 4, 6, 4,1 in $(x+y)^4$ etc. are called **binomial coefficients.** From the above examples we observe that the coefficients in these expansions are as follows.

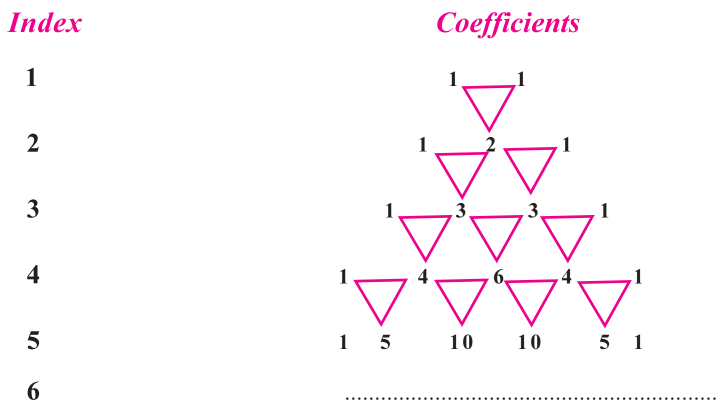


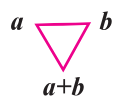
Fig. 6.1

From the above diagram we observe the following pattern of obtaining a row from the previous row from the second row onwards

(i) Each row begins and ends with 1 (one)

(ii) The n^{th} row has $(n+1)$ terms for any $n \in \mathbf{Z}^+$.

(iii) The other numbers (except the first and last) in a row are obtained by adding the two numbers in the previous row on either side of it. This addition is shown by means of the triangle in each row as follows



The diagram in Fig. 6.1 is called **Pascal triangle** which is named after its inventor, a French mathematician **Blaise Pascal** (1623-1662). But this was mentioned in a different form under a different name **Meru-Prastara** by the renowned Indian scientist **Pingala** in his book **Chanda Shastra** as early as 200 B.C.

The expansion of $(x+y)^n$ using multiplication, as shown at the beginning, becomes difficult as n increases. In this chapter, we derive the expansion of $(x+y)^n$ when n is a positive integer. This result is known as the **Binomial Theorem.** The coefficients of the terms $x^i y^j$ are called **Binomial Coefficients.** We study the properties of these binomial coefficients, give methods to find the middle term(s) and the numerically greatest term(s) in a binomial expansion. Also we outline (without Proof) the binomial theorem for the expansion of $(x+a)^n$ when n is a negative integer or any fraction. We find the coefficient of a particular index (power) of x in the expansion of $(x+a)^n$ (when n is an integer or a rational number). Finally, we find the approximate values of some irrational numbers using the binomial expansions.

6.1 Binomial Theorem for positive integral index

In the expansions of $(x+y)^n$, mentioned in the introduction, we observe the following points.

(i) As we proceed from left to right, the index of x decreases and the index of y increases by 1 at a time.

(ii) The coefficients in the expansion of

$(x+y)^1$ are ${}^1C_0, {}^1C_1,$

$(x+y)^2$ are ${}^2C_0, {}^2C_1, {}^2C_2$

$(x+y)^3$ are ${}^3C_0, {}^3C_1, {}^3C_2, {}^3C_3$

$(x+y)^4$ are ${}^4C_0, {}^4C_1, {}^4C_2, {}^4C_3, {}^4C_4$

From these observations we can easily guess the general formula for the expansion of $(x+y)^n$ for any positive integer n and verify the same using the principle of mathematical induction. This is shown in the following.

6.1.1 Theorem (Binomial Theorem)

Let n be a positive integer and x, a be real numbers, then

$$(x+a)^n = {}^nC_0 . x^n a^0 + {}^nC_1 . x^{n-1} . a^1 + {}^nC_2 . x^{n-2} . a^2 + \cdots + {}^nC_r . x^{n-r} . a^r + \cdots + {}^nC_n . x^0 . a^n$$

Proof : We prove this theorem by using the principle of mathematical induction (on n).

When $n=1, (x+a)^n = (x+a)^1 = x+a = {}^1C_0\, x^1 a^0 + {}^1C_1\, x^0 a^1$

Thus the theorem is true for $n=1$

Assume that the theorem is true for $n=k\geq 1$ (where k is a positive integer). That is

$$(x+a)^k = {}^kC_0 . x^k . a^0 + {}^kC_1 . x^{k-1} . a^1 + {}^kC_2 . x^{k-2} . a^2 + \cdots + {}^kC_r . x^{k-r} . a^r + \cdots + {}^kC_k . x^0 a^k$$

Now we prove that the theorem is true when $n = k+1$ also.

$$(x+a)^{k+1} = (x+a)(x+a)^k$$

$$= (x+a)({}^kC_0 . x^k . a^0 + {}^kC_1 . x^{k-1} . a^1 + {}^kC_2 . x^{k-2} . a^2 + \cdots$$
$$+ {}^kC_r . x^{k-r} . a^r + \cdots + {}^kC_k x^0 . a^k)$$

$$= {}^kC_0 x^{k+1} . a^0 + {}^kC_1 . x^k . a^1 . + {}^kC_2 x^{k-1} . a^2 + \cdots + {}^kC_r . x^{k-r+1} a^r + \cdots$$
$$+ {}^kC_k x^1 . a^k + {}^kC_0 . x^k a^1 + {}^kC_1 . x^{k-1} a^2 + \cdots$$
$$+ {}^kC_{r-1} . x^{k-r+1} . a^r + \cdots + {}^kC_{k-1} . x^1 a^k + {}^kC_k . x^0 . a^{k+1}$$

$$= {}^kC_0.x^{k+1}.a^0 + ({}^kC_1 + {}^kC_0).x^k.a^1 + ({}^kC_2 + {}^kC_1).x^{k-1}.a^2 + \cdots$$
$$+ ({}^kC_r + {}^kC_{r-1}).x^{k-r+1}.a^r + \cdots + ({}^kC_k + {}^kC_{k-1}).x^1.a^k$$
$$+ {}^kC_k.x^0.a^{k+1}$$

$$= {}^{(k+1)}C_0.x^{k+1}.\,a^0 + {}^{(k+1)}C_1.\,x^k.\,a^1 + {}^{(k+1)}C_2.\,x^{k-1}.\,a^2 + \cdots$$
$$+ {}^{(k+1)}C_r.\,x^{k+1-r}.\,a^r + \cdots + {}^{(k+1)}C_k.x^1.\,a^k + {}^{(k+1)}C_{(k+1)}.\,x^0.\,a^{k+1}$$

since ${}^kC_0 = 1 = {}^{k+1}C_0 \,.\, {}^kC_r + {}^kC_{r-1} = {}^{(k+1)}C_r$ for $1 \le r \le k$ and ${}^kC_k = 1 = {}^{(k+1)}C_{(k+1)}$

Therefore the theorem is true for $n = k+1$

Hence, by mathematical induction, it follows that the theorem is true of all positive integers n.

From the above theorem we make the following observations.

6.1.2 Note

Let n be a positive integer and x, a be real numbers, then

(i) $$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$$

(ii) The expansion of $(x+a)^n$ has $(n+1)$ terms

(iii) The r^{th} term in the expansion of $(x+a)^n$, which is denoted by T_r, is given by

$$T_r = {}^nC_{r-1}.\, x^{n-r+1}.a^{r-1} \quad \textit{for} \quad 1 \le r \le n+1$$

In the following we define the general term in the expansion of $(x+a)^n$

6.1.3 Definition

In the expansion of $(x+a)^n$ *the* $(r+1)^{th}$ *term is called* ***the general term*** *and it is given by*

$$T_{r+1} = {}^nC_r\,.\,x^{n-r}.\,a^r \quad \text{for} \quad 0 \le r \le n$$

In the expansion of $(x+a)^n$, a is either a positive or a negative real number and hence there is no need to give the expansion of $(x-a)^n$ separately. But still it will be useful to the reader to have the expansion of $(x-a)^n$ clearly. It is given in the next page.

6.1.4 Note

On replacing a by $-a$ in the expansion of $(x+a)^n$ given in Theorem 6.1.1, we get

$$(x-a)^n = {}^nC_0.x^n(-a)^0 + {}^nC_1.x^{n-1}(-a)^1 + {}^nC_2.x^{n-2}(-a)^2 + \cdots + {}^nC_r.x^{n-r}(-a)^r + \cdots + {}^nC_n.x^0(-a)^n$$

$$= {}^nC_0.x^n a^0 - {}^nC_1.x^{n-1}.a^1 . + {}^nC_2.x^{n-2}.a^2 - \cdots + (-1)^r.{}^nC_r.x^{n-r}.a^r + \ldots + (-1)^n.{}^nC_n.x^0 a^n.$$

In this expansion, the general term T_{r+1} is given by

$$\boxed{T_{r+1} = (-1)^r.{}^nC_r.x^{n-r}.a^r} \quad for \ \ 0 \le r \le n$$

Now we give some examples in which we apply the binomial Theorem (6.1.1).

6.1.5 Examples

1. Example : Write the expansion of $(2a+3b)^6$.

Write $2a = x$ and $3b = y$. Then

$$(2a+3b)^6 = (x+a)^6 = {}^6C_0\, x^6 a^0 + {}^6C_1\, x^5.a^1 + {}^6C_2\, x^4.a^2 + {}^6C_3\, x^3.a^3 + {}^6C_4\, x^2.a^4 + {}^6C_5\, x^1.a^5 + {}^6C_6\, x^0 a^6$$

$$= {}^6C_0\,(2a)^6(3b)^0 + {}^6C_1\,(2a)^5(3b)^1 + {}^6C_2\,(2a)^4(3b)^2 + {}^6C_3\,(2a)^3(3b)^3 + {}^6C_4\,(2a)^2(3b)^4 + {}^6C_5\,(2a)^1(3b)^5 + {}^6C_6\,(2a)^0(3b)^6$$

$$= 2^6 a^6 + 6.2^5.3.a^5 b + 15.2^4.3^2.a^4 b^2 + 20.2^3.3^3.a^3 b^3 + 15.2^2.3^4.a^2 b^4 + 6.2.3^5.a.b^5 + 3^6.b^6$$

$$= 64a^6 + 576\, a^5 b + 2160\, a^4 b^2 + 4320 a^3 b^3 + 4860 a^2 b^4 + 2916 a.b^5 + 729 b^6.$$

2. Example : Find the 5^{th} term in the expansion of $(3x-4y)^7$.

Write $b = 3x$ and $a = 4y$. So that

$$(3x-4y)^7 = (b-a)^7.$$

Now, the general term T_{r+1} in $(3x-4y)^7 = (b-a)^7$ is given by

$$T_{r+1} = {}^7C_r.b^{7-r}(-a)^r$$

$$= {}^7C_r\,(3x)^{7-r}(-4y)^r.$$

To find 5^{th} term, put $r = 4$. we get

$$T_5 = {}^7C_4\,(3x)^3(-4y)^4$$

$$= 35 \times 27 x^3 \times 256 y^4$$

$$= 35 \times 27 \times 256 \times x^3.\ y^4$$

$$= 241920 x^3\, y^4.$$

3. Example : Find the 4^{th} term from the end in the expansion of $(2a+5b)^8$

The expansion has 9 terms so that the fourth term from the end is 6^{th} term from the beginning. In the expansion of $(2a+5b)^8$,

$$T_{r+1} = {}^8C_r\,(2a)^{8-r}.(5b)^r \quad (1 \le r \le 8)$$

Substituting $r=5$, we get

$$T_6 = {}^8C_5.(2a)^{8-5}.(5b)^5 = {}^8C_5.2^3.5^5a^3b^5.$$

Thus the 4^{th} term from the end is ${}^8C_5.2^3.5^5a^3b^5$.

6.1.6 Theorem (Trinomial Expansion)

Let $n \in \mathbf{N}$ and $a,b,c \in \mathbf{R}$. Then $(a+b+c)^n$ *can be expanded using the binomial theorem by taking* a *as the first term and (b + c) as the second term as follows.*

$$(a+b+c)^n = (a+(b+c))^n = \sum_{r=0}^{n} {}^nC_r\,.\,a^{n-r}.(b+c)^r$$

Now, for $0 \le r \le n$, *the expansion of* $(b+c)^r$ *contains (r + 1) terms, Hence the number of terms in the expansion of* $(a+b+c)^n$

$$= \sum_{r=0}^{n}(r+1) = \frac{(n+1)(n+2)}{2}.$$

Another form of the expansion is

$$(a+b+c)^n = \sum_{\substack{o \le p,q,r \le n \\ p+q+r=n}} \frac{n!}{p!q!r!}\,.\,a^p.\,b^q.\,c^r$$

In the above expansion, the summation is taken over all ordered triplets (p,q,r) of non-negative integers whose sum is n. (*This is called as **'Trinomial Expansion'***).

Now we define the middle term(s) in the expansion of $(x+a)^n$.

6.1.7 Definition

(i) *For any positive integer n, since the expansion on of* $(x+a)^n$ *contains* $(n+1)$ *terms, if n is even then the number of terms in the expansion is odd. Hence there is only one middle term, which is the* $\left(\frac{n}{2}+1\right)^{th}$ *term and*

$$T_{\frac{n}{2}+1} = {}^nC_{\left(\frac{n}{2}\right)}.x^{\frac{n}{2}}.a^{\frac{n}{2}}$$

(ii) *For any positive integer n, since expansion of* $(x + a)^n$ *contains* $(n + 1)$ *terms, if n is odd then the number of terms in the expansion is even. Hence, there are two middle terms and they are* $\left(\frac{n+1}{2}\right)^{th}$ *and* $\left(\frac{n+1}{2}+1\right)^{th}$ *terms. These terms are given by*

$$T_{\frac{n+1}{2}} = {}^nC_{\left(\frac{n-1}{2}\right)}.x^{\frac{n+1}{2}}.a^{\frac{n-1}{2}}$$

$$\text{and}\quad T_{\frac{n+3}{2}} = {}^nC_{\left(\frac{n+1}{2}\right)}.x^{\frac{n-1}{2}}.a^{\frac{n+1}{2}}$$

6.1.8 Example

Find the middle term(s) of the following expansions.

(i) $(3a-5b)^6$ (ii) $(2x+3y)^7$

(i) The expansion of $(3a-5b)^6$ has 7 (odd number) terms. Hence there is only one middle term and it is

$$T_4 = {}^6C_3.(3a)^3.(-5b)^3 = -{}^6C_3.\,3^3.\,5^3.\,a^3.\,b^3.$$

(ii) The expansion of $(2x+3y)^7$ has 8 (even number) terms. Hence there are two middle terms. They are T_4 and T_5. Now

$$T_4 = {}^7C_3\,(2x)^4\,(3y)^3 = {}^7C_3.\,2^4.3^3.\,x^4.\,y^3.$$

$$T_5 = {}^7C_4\,(2x)^3\,(3y)^4 = {}^7C_4.2^3.\,3^4.x^3.y^4.$$

Next we discuss certain properties of the coefficients in the binomial expansion.

6.1.9 Definition

The coefficients in the binomial expansion (Theorem 6.1.1) are

$${}^nC_0, {}^nC_1, {}^nC_2,..., {}^nC_r,..., {}^nC_n$$

These coefficients are called the ***binomial coefficients*** *(corresponding to n). When n is fixed, these coefficients are usually denoted by*

$C_0, C_1, C_2,..., C_r,..., C_n$ *respectively.*

In this notation, $C_r = C_{n-r}$, since ${}^nC_r = {}^nC_{n-r}$ *for* $0 \le r \le n$.

That is, $C_o = C_n$, $C_1 = C_{n-1}$, $C_2 = C_{n-2}$ etc.

6.1.10 Note

The binomial expansion of $(1+x)^n$ is given by

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$$

This expansion is called **the standard binomial expansion**. Now we prove certain important properties of binomial coefficients in the following.

6.1.11 Theorem

With the standard notation, if n is a positive integer, then

(i) $C_0 + C_1 + C_2 + \cdots + C_n = 2^n$

(ii) (a) $C_0 + C_2 + C_4 + \cdots + C_n = 2^{n-1}$ *if n is even*

(b) $C_0 + C_2 + C_4 + \cdots + C_{n-1} = 2^{n-1}$ *if n is odd*

(iii) (a) $C_1 + C_3 + C_5 + \cdots + C_{n-1} = 2^{n-1}$ *if n is even*

(b) $C_1 + C_3 + C_5 + \cdots + C_n = 2^{n-1}$ *if n is odd*

Proof : From the standard binomial expansion (see Note. 6.1.10), we have

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$$

$$= C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n \qquad \text{... (1)}$$

(i) on substituting $x = 1$ in (1), we get

$$C_0 + C_1 + C_2 + \cdots + C_n = 2^n \qquad \text{... (2)}$$

(ii) on substituting $x = -1$ in (1), we get

$$C_0 - C_1 + C_2 - C_3 + \cdots + (-1)^n . C_n = 0 \qquad \text{... (3)}$$

(iii) on adding (2) and (3), we get

$$2C_0 + 2C_2 + \cdots + \left(1 + (-1)^n\right)C_n = 2^n$$

$$\text{i.e, } C_0 + C_2 + \cdots + \left(\frac{1+(-1)^n}{2}\right)C_n = 2^{n-1}$$

Thus (a) $C_0 + C_2 + \cdots + C_n = 2^{n-1}$ if n is even

(b) $C_0 + C_2 + \cdots + C_{n-1} = 2^{n-1}$ if n is odd

(iv) on subtracting (3) from (2), we get

$$2C_1 + 2C_3 + \ldots + (1 - (-1)^n)\, C_n = 2^n$$

As above we get

(a) $C_1 + C_3 + \ldots + C_{n-1} = 2^{n-1}$ if n is even

(b) $C_1 + C_3 + \ldots + C_n = 2^{n-1}$ if n is odd

6.1.12 Example

Prove that $C_0 + 3.C_1 + 5 . C_2 + \cdots + (2n+1). C_n = (2n+2). 2^{n-1}$

Solution : Write $S = C_0 + 3C_1 + 5C_2 + \cdots + (2n+1)\, C_n$ ... (1)

By writing the terms in (1) in the reverse order, we get

$$S = (2n+1). C_n + (2n-1).C_{n-1} + (2n-3).C_{n-2} + \cdots + 3\, C_1 + 1\, C_0$$
$$= (2n+1). C_0 + (2n-1).C_1 + (2n-3).C_2 + \cdots + 3\, C_{n-1} + C_n . \quad \text{... (2)}$$

(Since $C_{n-r} = C_r$)

On adding (1) and (2), We get

$$2S = (2n+2). C_0 + (2n+2).C_1 + \cdots + (2n+2).C_{n-1} + (2n+2)\, C_n$$
$$= (2n+2)\,(C_0 + C_1 + C_2 + \cdots + C_n)$$
$$= (2n+2).2^n.$$

Therefore, $S = (2n+2)\,2^{n-1}$.

6.1.13 Note

In the above example observe that the coefficients of $C_0, C_1, C_2, ..., C_n$ are respectively, 1, 3, 5, ... $(2n+1)$. These coefficients are in A.P.. The above problem is generalized in the following and it is proved in a similar method.

6.1.14 Examples

1. Example : Prove that, for any real numbers *a, d,*

$$\boxed{a.C_0 + (a+d).C_1 + (a+2d).C_2 + \cdots + (a+nd). C_n = (2a+nd).2^{n-1}}$$

Solution : Write $S = a.C_0 + (a+d).C_1 + (a+2d).C_2 + \cdots + (a+nd)\,.C_n$ ... (1)

By writing the terms in the R.H.S. of (1) in the reverse order as done in example 6.1.12, we get

$$S = (a+nd).C_n + (a+\overline{n-1}d)\,.C_{n-1} + (a+\overline{n-2}d)\,.\,C_{n-2} + \cdots + a.C_o .$$

Using the fact that $C_r = C_{n-r}$ for $0 \le r \le n$, we get

$$S = (a+nd).C_0 + (a+\overline{n-1}d).C_1 + (a+\overline{n-2}d)\,.C_2 + \cdots + a.C_n . \quad \text{... (2)}$$

On adding (1) and (2), we get

$$\begin{aligned} 2S &= (2a+nd).C_0 + (2a+nd).C_1 + (2a+nd).C_2 + \cdots + (2a+nd)\,C_n \\ &= (2a+nd)\,(C_0 + C_1 + C_2 + \cdots + C_n) \\ &= (2a+nd).2^n \quad \text{from theorem} \quad 6.1.11\,(i). \end{aligned}$$

Therefore $S = (2a+nd).\,2^{n-1}$.

2. Example

If n is a positive integer, then we prove that

(i) $\boxed{\sum_{r=1}^{n} r.C_r = n.2^{n-1}}$

(ii) $\boxed{\sum_{r=2}^{n} r(r-1).C_r = n(n-1).2^{n-2}}$

(iii) $\boxed{\sum_{r=1}^{n} r^2.\,C_r = n(n+1).2^{n-2}}$

Solution

(i) **First Method**

L.H.S. $= 0.C_o + 1.C_1 + 2.C_2 + 3.C_3 + \cdots + n.C_n$

In this, the coefficients $0, 1, 2, ..., n$ of $C_o, C_1, C_2, ..., C_n$ are in A.P with first term a=0 and common difference d=1. Hence, by Example 1, above,

L.H.S. $= (2a+nd).\,2^{n-1} = n.\,2^{n-1}$

Second Method : We have

$(1+x)^n = C_o + C_1 x + C_2 x^2 + \cdots + C_n x^n$

On differentiating both sides with respect to x, we get

$n(1+x)^{n-1} = C_1 + C_2(2x) + C_3(3x^2) + ... + C_n(nx^{n-1})$.

Now put $x = 1$. We get

$n.2^{n-1} = C_1 + 2.C_2 + 3.C_3 + ... + n.C_n.$ Thus $\sum_{r=1}^{n} r.C_r = n.2^{n-1}$

(ii) We have

$(1+x)^n = C_o + C_1.\,x + C_2.\,x^2 + C_3.\,x^3 + \cdots + C_n.\,x^n.$

On differentiating both sides with respect to x, we get

$$n(1+x)^{n-1} = C_1 + 2.\,C_2.\,x + 3.C_3.\,x^2 + \cdots + n.\,C_n.\,x^{n-1}$$

Again, on differentiating both sides with respect to x, we get

$$n(n-1).(1+x)^{n-2} = 2.1.\,C_2. + 3.\,2.\,C_3.\,x + \cdots + n(n-1).\,C_n.\,x^{n-2}.$$

Now, on substituting $x = 1$, we get

$$n(n-1).\,2^{n-2} = 2.1.\,C_2 + 3.\,2.\,C_3 + \cdots + n.(n-1).C_n$$

Thus, $\sum_{r=2}^{n} r(r-1).\,C_r = n(n-1).\,2^{n-2}$

(iii) $\sum_{r=1}^{n} r^2.C_r = \sum_{r=1}^{n}(r(r-1)+r).\,C_r = \sum_{r=1}^{n} r(r-1).\,C_r + \sum_{r=1}^{n} r.C_r$

$= n(n-1).2^{n-2} + n.2^{n-1}$ (by (i) and (ii) above)

$= (n-1+2).n.2^{n-2} = n(n+1).2^{n-2}.$

In definition 6.1.7, we have defined the concept of middle term(s) of a binomial expansion. Now we define the concept of numerically greatest term in a binomial expansion and discuss methods of finding it. First we give the following.

6.1.15 Definition (Integral Part and Fractional Part)

We recall from lesson 1 of I (A) the definition of the integral part function. If x is any real number, then there always exists an integer n such that $n \le x < n+1$. In other words, there exists a greatest integer n such that $n \le x$. This integer n is called the integral part of the real number x and it is denoted by $[x]$. The real number $x-[x]$ is called the fractional part of x and it is usually denoted by $\{x\}$.

6.1.16 Example

(i) If $x = \dfrac{17}{3}$, then $[x] = 5$ and $\{x\} = \dfrac{17}{3} - 5 = \dfrac{2}{3}$

(ii) If $x = -\dfrac{14}{5}$, then $[x] = -3$ and $\{x\} = \dfrac{-14}{5} - (-3) = \dfrac{1}{5}$

(iii) If $x = \sqrt{46}$, then $[x] = 6$ and $\{x\} = \sqrt{46} - 6$

(iv) If $x = 10$, then $[x] = 10$ and $\{x\} = 0$

Note that, for any real number x, $0 \le \{x\} < 1$.

6.1.17 Definition (Numerically greatest term)

In the binomial expansion of $(1+x)^n$, *the* r^{th} *term* T_r *is called the* ***numerically greatest term*** *of the expansion of* $(1+x)^n$ *if*

$$|T_k| \le |T_r| \text{ for } 1 \le k \le n+1$$

In other words the absolute value of T_r is greatest among the absolute values of all terms of the expansion. In the following theorem, we explain a method to find the numerically greatest term in the expansion of $(1+x)^n$.

6.1.18 Theorem

Let $n \in \mathbf{N}$, $x \in \mathbf{R}-\{0\}$ *and* $m = \left[\dfrac{(n+1)|x|}{1+|x|}\right]$ *(here, for any* $t \in \mathbf{R}$, $[t]$ *is the integral part of* t*).*

(i) *If* $\dfrac{(n+1)|x|}{1+|x|}$ *is not an integer, then* T_{m+1} *is the numerically greatest term in the binomial expansion of* $(1+x)^n$.

(ii) *If* $\dfrac{(n+1)|x|}{1+|x|}$ *is an integer, then* $|T_m| = |T_{m+1}|$ *and hence* T_m and T_{m+1} *are numerically greatest terms in the binomial expansion of* $(1+x)^n$.

Proof : In the binomial expansion of $(1+x)^n$, for $1 \le r < n$,

$T_{r+1} = {}^nC_r \,.\, x^r$ and $T_r = {}^nC_{(r-1)} x^{(r-1)}$. Since $x \ne 0$, T_{r+1} and T_r are non zero and hence

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r . x^r}{{}^nC_{(r-1)} x^{(r-1)}} = \frac{n!}{(n-r)!r!} \times \frac{(n-r+1)!(r-1)!}{n!} . x = \frac{n-r+1}{r} . x$$

Therefore, $\dfrac{|T_{r+1}|}{|T_r|} = \left|\dfrac{n-r+1}{r} . x\right| = \dfrac{n-r+1}{r} . |x|$ (Since $n-r+1$ and r are positive integers)

Now, $|T_{r+1}| \ge |T_r| \Leftrightarrow \left|\dfrac{T_{r+1}}{T_r}\right| \ge 1$.

i.e., $|T_{r+1}| > |T_r| \Leftrightarrow \left|\dfrac{T_{r+1}}{T_r}\right| > 1$ and $|T_{r+1}| = |T_r| \Leftrightarrow \left|\dfrac{T_{r+1}}{T_r}\right| = 1$

$$\therefore \ |T_{r+1}| \ge |T_r| \Leftrightarrow \left(\frac{n-r+1}{r}\right)|x| \ge 1$$

$$\Leftrightarrow \quad \frac{n+1}{r} - 1 \geq \frac{1}{|x|} \qquad (\text{since } |x| > 0)$$

$$\Leftrightarrow \quad \frac{n+1}{r} \geq \frac{1}{|x|} + 1 = \frac{1+|x|}{|x|}$$

$$\Leftrightarrow \quad r \leq \frac{(n+1)|x|}{1+|x|} \qquad \dots (1)$$

(i) Suppose $\frac{(n+1)|x|}{1+|x|}$ is not an integer

Then $m < \frac{(n+1)|x|}{1+|x|} < m+1$ $\left(\text{recall that, by hypothesis, } m = \left[\frac{(n+1)|x|}{1+|x|}\right]\right)$

For $1 \leq r \leq m,$ *we have* $r < \frac{(n+1)|x|}{1+|x|}$. Hence, from (1), we get that

$$|T_{r+1}| > |T_r|$$

That is, $|T_1| < |T_2| < \dots\dots < |T_{m+1}|$ ... (2)

For $m+1 \leq r \leq n,$ *we have* $r > \frac{(n+1)|x|}{1+|x|}$. Hence, from (1), we get that

$$|T_{r+1}| < |T_r|$$

That is, $|T_{n+1}| < |T_n| < \dots < |T_{m+2}| < |T_{m+1}|$ ... (3)

From (2) and (3), we get that

$$|T_{m+1}| > |T_r| \quad \text{for} \quad 1 \leq r \leq n+1 \quad \text{and} \quad r \neq m+1.$$

Hence, T_{m+1} is the numerically greatest term in the binomial expansion of $(1+x)^n$.

(ii) Suppose $\frac{(n+1)|x|}{1+|x|}$ is an integer

Then $m = \frac{(n+1)|x|}{1+|x|}$. From (1), as above we get

$$|T_{r+1}| > |T_r| \quad \text{for} \quad 1 \leq r \leq m-1$$

That is, $|T_1| < |T_2| < \cdots < |T_m|$ ... (4)

and $|T_{r+1}| < |T_r|$ for $m+1 \leq r \leq n.$

That is $|T_{n+1}| < |T_n| < \dots < |T_{m+2}| < |T_{m+1}|$ ... (5)

Also when $r = m$ we get, from (1), that

$$|T_m| = |T_{m+1}| \quad \text{... (6)}$$

From (4), (5) and (6) we conclude that

$$|T_m| = |T_{m+1}| > |T_r| \quad \text{for all} \quad r \in \{1,2,3,...,n+1\} - \{m, m+1\}$$

Hence, T_m and T_{m+1} are numerically greatest terms in the binomial expansion of $(1+x)^n$.

6.1.19 Corollary (Largest binomial coefficient)

The largest binomial coefficient(s) among ${}^nC_0, {}^nC_1, ..., {}^nC_n$ is (are)

(i) ${}^nC_{\left(\frac{n}{2}\right)}$ if n is an even integer

(ii) ${}^nC_{\left(\frac{n-1}{2}\right)} = {}^nC_{\left(\frac{n+1}{2}\right)}$ if n is an odd integer

Proof : On substituting $x=1$ in Theorem 6.1.18, we get

$$(1+1)^n = {}^nC_o + {}^nC_1 + {}^nC_2 + ... + {}^nC_n \quad \text{... (1)}$$

(i) If n is even, then $m = \left[\frac{(n+1)|x|}{1+|x|}\right] = \left[\frac{n+1}{2}\right] = \frac{n}{2}$. Hence $T_{m+1} = {}^nC_{\frac{n}{2}}$ is the greatest among all terms in (1). (That is, among all binomial coefficients)

(ii) If n is odd then $\frac{(n+1)|x|}{1+|x|} = \frac{n+1}{2}$ is an integer and $m = \frac{n+1}{2}$. Hence, by Theorem 6.1.18 $T_m = {}^nC_{\frac{n-1}{2}}$ and $T_{m+1} = {}^nC_{\frac{n+1}{2}}$ are greatest binomial coefficients. Observe that ${}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$.

Note : From the above corollary, it can be observed that, for any positive integer n, the greatest binomial coefficient among ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ is ${}^nC_r$ where $r = \left[\frac{n}{2}\right]$.

6.1.20 Example

Let us find the numerically greatest term in the binomial expansion of $(1-5x)^{12}$ when $x = \frac{2}{3}$.

On comparing $(1-5x)^{12}$ with $(1+x)^n$, we get $n = 12$, $x = -5x = \frac{-10}{3}$ so that

$$\frac{(n+1)|x|}{1+|x|} = \frac{13.\left|\frac{-10}{3}\right|}{1+\left|\frac{-10}{3}\right|} = \frac{130}{13} = 10 \quad \text{(an integer)}$$

Hence, by (ii) of Theorem 6.1.18, we get that the numerically greatest term in the binomial expansion of $(1-5x)^{12}$ are $T_m = T_{10}$ and $T_{m+1} = T_{11}$. They are given by

$$T_{10} = {}^{12}C_9.(-5x)^9 = -{}^{12}C_9.\left(\frac{10}{3}\right)^9$$

and $T_{11} = {}^{12}C_{10}.(-5x)^{10} = {}^{12}C_{10}.\left(\frac{10}{3}\right)^{10}$. Note that $|T_{10}| = |T_{11}| = 220\left(\frac{10}{3}\right)^9$.

6.1.21 Note

To find the numerically greatest term(s) in the binomial expansion of $(a+x)^n$ (where a, x are non zero real numbers) we write $(a+x)^n = a^n(1+\frac{x}{a})^n$ and then find the numerically greatest term(s) in the binomial expansion of $(1+\frac{x}{a})^n$ using Theorem 6.1.18. Finally, we multiply it (them) by $|a^n|$ to get the numerically greatest term(s) in the expansion of $(a+x)^n$.

6.1.22 Example

Let us compute the numerically greatest term(s) in the expansion of $(3x-5y)^n$ when $x=\frac{3}{4}$, $y=\frac{2}{7}$ and $n=17$

First write $(3x-5y)^n = (3x-5y)^{17} = (3x)^{17}(1-\frac{5y}{3x})^{17}$

on comparing $(1-\frac{5y}{3x})^{17}$ with $(1+X)^n$, we get $n=17$, $X = \frac{-5y}{3x} = \frac{-10}{7}.\frac{4}{9} = -\frac{40}{63}$.

Now, $\frac{(n+1)|X|}{1+|X|} = \frac{18.\frac{40}{63}}{1+\frac{40}{63}} = \frac{720}{103}$, which is not an integer.

Therefore, $m = \left[\frac{720}{103}\right] = 6$.

Hence, by Theorem 6.1.18, the numerically greatest term in the expansion of $\left(1-\frac{5y}{3x}\right)^{17}$ is the 7^{th} term. Let T_r denote the r^{th} term in the expansion of $(3x-5y)^{17} = (3x)^{17}\left(1-\frac{5y}{3x}\right)^{17}$. Then T_7 is the numerically greatest term in it and

$$T_7 = (3x)^{17} \cdot {}^{17}C_6\left(\frac{-5y}{3x}\right)^6 = {}^{17}C_6\,(5y)^6.(3x)^{11}$$

$$= {}^{17}C_6\left(\frac{10}{7}\right)^6 \cdot \left(\frac{9}{4}\right)^{11} = {}^{17}C_6 \cdot \frac{5^6.9^{11}}{7^6.4^8}.$$

6.1.23 Solved Problems

1. Problem : *Find the largest binomial coefficient(s) in the expansion of*

(i) $(1+x)^{19}$ (ii) $(1+x)^{24}$

Solutions

(i) Here $n=19$, an odd integer. Therefore, by corollary 6.1.19, the largest binomial coefficients are ${}^nC_{\left(\frac{n-1}{2}\right)}$ and ${}^nC_{\left(\frac{n+1}{2}\right)}$, that is ${}^{19}C_9$ and ${}^{19}C_{10}$. (Note that ${}^{19}C_9 = {}^{19}C_{10}$).

(ii) Here $n=24$ is an even integer. Hence there is only one largest binomial coefficient, that is, ${}^nC_{\left(\frac{n}{2}\right)} = {}^{24}C_{12}$.

2. Problem : *If* ${}^{22}C_r$ *is the largest binomial coefficient in the expansion of* $(1+x)^{22}$ *find the value of* ${}^{13}C_r$.

Solution : Here $n=22$ is an even integer. Therefore, there is only one largest binomial coefficient and it is ${}^nC_{\left(\frac{n}{2}\right)} = {}^{22}C_{11} = {}^{22}C_r$ (given). Hence $r = 11$. Now

$${}^{13}C_r = {}^{13}C_{11} = {}^{13}C_2 = \frac{13\times 12}{1\times 2} = 78.$$

3. Problem : *Find the 7^{th} term in the expansion of* $\left(\frac{4}{x^3}+\frac{x^2}{2}\right)^{14}$.

Solution : The general term in the expansion of $(X+a)^n$ is given by

$$T_{r+1} = {}^nC_r \,.\, X^{n-r} \,.\, a^r$$

put $X = \frac{4}{x^3}$, $a = \frac{x^2}{2}$ and $r = 6$, we get the 7^{th} term in the expansion of $\left(\dfrac{4}{x^3}+\dfrac{x^2}{2}\right)^{14}$ as

$$T_7 = {}^{14}C_6\left(\frac{4}{x^3}\right)^{14-6}\left(\frac{x^2}{2}\right)^6 = {}^{14}C_6 . \frac{4^8}{2^6} . \frac{x^{12}}{x^{24}} = {}^{14}C_6 . \frac{4^5}{x^{12}}.$$

4. Problem : *Find the 3rd term from the end in the expansion of* $\left(x^{\frac{-2}{3}}-\dfrac{3}{x^2}\right)^8$.

Solution : Comparing the given expansion with $(X+a)^n$, we get

$$X = x^{\frac{-2}{3}}, \quad a = \frac{-3}{x^2}, \quad n = 8.$$

The expansion has $(n+1)=9$ terms. Hence the 3^{rd} term from the end is 7^{th} term from the beginning and

$$T_7 = {}^nC_6 . X^{n-6} . a^6$$

$$= {}^8C_6\left(x^{\frac{-2}{3}}\right)^{8-6} . \left(\frac{-3}{x^2}\right)^6 = 28\times x^{\frac{-4}{3}} . \frac{3^6}{x^{12}} = \frac{28\times 3^6}{x^{\frac{40}{3}}}. \quad \ldots (1)$$

5. Problem : *Find the coefficients of* x^9 and x^{10} *in the expansion of* $\left(2x^2-\dfrac{1}{x}\right)^{20}$.

Solution : Write $X = 2x^2,\ a = \dfrac{-1}{x},\ n = 20$. Then $\left(2x^2-\dfrac{1}{x}\right)^{20} = (X+a)^n$.

In this expansion, general term is

$$T_{r+1} = {}^nC_r . X^{n-r} . a^r$$

$$= {}^{20}C_r . (2x^2)^{20-r} . \left(\frac{-1}{x}\right)^r$$

$$= (-1)^r . {}^{20}C_r . 2^{20-r} . x^{40-3r}.$$

To find the coefficient of x^9 put $40-3r=9$. Then we get $r=\dfrac{31}{3}$.

Since r is a positive integer this is not possible. This means that the expansion of $\left(2x^2-\dfrac{1}{x}\right)^{20}$ doesn't posess the term containing x^9. This means that the coefficient of x^9 in the expansion of $\left(2x^2-\dfrac{1}{x}\right)^{20}$ is 0.

To find the coefficient of x^{10} put $40-3r=10$. We get $r=10$.
Now, on substituting $r=10$ in (1), we get that

$$T_{11} = (-1)^{10} . {}^{20}C_{10} . 2^{10} . x^{10}$$

Hence, the coefficient of x^{10} in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{20}$ is ${}^{20}C_{10}.2^{10}$.

6. Problem : *Find the term independent of x (that is , the constant term) in the expansion of* $\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$.

Solution : Take $X = \sqrt{\frac{x}{3}}$, $a = \frac{3}{2x^2}$, $n = 10$. Then $\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10} = (X+a)^n$.

The general term in this expansion is

$$T_{r+1} = {}^{n}C_r \, X^{n-r} . a^r = {}^{10}C_r . \left(\sqrt{\frac{x}{3}}\right)^{10-r} . \left(\frac{3}{2x^2}\right)^r$$

$$= {}^{10}C_r \frac{3^{\frac{3r}{2}-5}}{2^r} . x^{5-\frac{5r}{2}}$$

To find the term independent of x (i.e., the coefficient of x^0),

Put $5 - \frac{5r}{2} = 0$. Then $r = 2$

Therefore, the term independent of x in the given expansion is

$$T_3 = {}^{10}C_2 . \frac{3^{-2}}{2^2} \; x^0 = \frac{45}{36} = \frac{5}{4}.$$

7. Problem : *If the coefficient of x^{10} in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is equal to the coefficeint of x^{-10} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$, find the relation between a and b where a and b are real numbers.*

Solution

Step 1 : The general term in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is

$$T_{r+1} = {}^{11}C_r . (ax^2)^{11-r} . \left(\frac{1}{bx}\right)^r = {}^{11}C_r . \frac{a^{11-r}}{b^r} . x^{22-3r}$$

To find the coefficient of x^{10} in this expansion, we should consider $22-3r=10$ or $r=4$

Hence, the coefficient of x^{10} in the expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$ is

$$= {}^{11}C_4 \cdot \frac{a^7}{b^4}. \qquad \dots (1)$$

Step 2 : The general term in the expansion of $\left(ax-\frac{1}{bx^2}\right)^{11}$ is

$$T_{r+1} = {}^{11}C_r \cdot (ax)^{11-r} \cdot \left(\frac{-1}{bx^2}\right)^r$$

$$= (-1)^r \cdot {}^{11}C_r \cdot \frac{a^{11-r}}{b^r} \cdot x^{11-3r}$$

To find the coefficient of x^{-10} in this expansion we should consider $11-3r = -10$ or $r=7$. Thus the coefficent of x^{-10} in the expansion of $\left(ax-\frac{1}{bx^2}\right)^{11}$ is

$$= (-1)^7 \cdot {}^{11}C_7 \cdot \frac{a^4}{b^7} \qquad \dots (2)$$

From the hypothesis, these coefficients are equal. Hence from (1) and (2), we get

$${}^{11}C_4 \cdot \frac{a^7}{b^4} = -{}^{11}C_7 \cdot \frac{a^4}{b^7}$$

i.e., $a^3 = -\frac{1}{b^3}$ (since ${}^{11}C_4 = {}^{11}C_7$)

i.e., $a^3b^3 = -1$

i.e., $ab = -1$ (since a, b are real).

8. Problem : *If the k^{th} term is the middle term in the expansion of* $\left(x^2-\frac{1}{2x}\right)^{20}$, *find* T_k *and* T_{k+3}.

Solution : The general term in the expansion of $\left(x^2-\frac{1}{2x}\right)^{20}$ is given by

$$T_{r+1} = {}^{20}C_r \cdot (x^2)^{20-r} \cdot \left(\frac{-1}{2x}\right)^r \qquad \dots (1)$$

The given expansion has 21 (odd number) terms. Hence T_{11} is the only middle term.

Thus $k = 11$.

On putting $r = 10$ in (1) we get

$$T_{11} = {}^{20}C_{10} \cdot x^{20} \cdot \left(\frac{-1}{2x}\right)^{10} = {}^{20}C_{10} \cdot \frac{x^{10}}{2^{10}}$$

Now to find $T_{k+3} = T_{11+3} = T_{14}$, put $r = 13$ in (1). Then

$$T_{14} = {}^{20}C_{13} \cdot (x^2)^7 \cdot \left(\frac{-1}{2x}\right)^{13} = -{}^{20}C_{13} \cdot \frac{1}{2^{13}} \cdot x.$$

9. Problem : *If the coefficients of* $(2r+4)^{th}$ *and* $(r-2)^{nd}$ *terms in the expansion of* $(1+x)^{18}$ *are equal, find r.*

Solution

The r^{th} term in the given expansion of $(1+x)^{18}$ is

$$T_r = {}^{18}C_{(r-1)} \cdot x^{r-1}$$

Thus, the coefficient of r^{th} term $= {}^{18}C_{r-1}$.

Given that the coefficient of $(2r+4)^{th}$ term = the coefficient of $(r-2)^{nd}$ term.

That is, ${}^{18}C_{2r+3} = {}^{18}C_{r-3}$

$$\Rightarrow 2r+3 = r-3 \text{ or } (2r+3)+(r-3) = 18$$
$$\Rightarrow r = -6 \text{ or } r = 6$$

Since r is a positive integer we get $r = 6$.

10. Problem : *Prove that* $2.C_0 + 7.C_1 + 12.C_2 + \cdots + (5n+2)C_n = (5n+4)2^{n-1}$

Solution

First method : The coefficients of $C_0, C_1, C_2, \ldots, C_n$ in L.H.S. are 2, 7, 12, ..., $(5n+2)$ which are in A.P. with first term $a = 2$ and common difference $d = 5$.

Hence from example 6.1.14(1), we get that

$$2.C_0 + 7.C_1 + 12.C_2 + \cdots + (5n+2).C_n = (2a+nd).2^{n-1}$$
$$= (4+5n)2^{n-1}$$

Second method : The general term $((r+1)^{\text{th}}$ term) in L.H.S. $=(5r+2).C_r$ Therefore,

$$\begin{aligned}
2.C_0+7.C_1+12.C_2+\cdots+(5n+2).C_n &= \sum_{r=0}^{n}(5r+2).C_r\\
&= 5\sum_{r=0}^{n} r.C_r + 2\sum_{r=0}^{n} C_r\\
&= 5\sum_{r=1}^{n} r.C_r + 2\sum_{r=0}^{n} C_r\\
&= 5.n2^{n-1}+2.2^n \text{ (from example 6.1.14,(2))}\\
&= (5n+4).2^{n-1}.
\end{aligned}$$

11. Problem : *Prove that*

(i) $C_0+3.C_1+3^2.C_2+\cdots+3^n.C_n=4^n$

(ii) $\dfrac{C_1}{C_0}+2.\dfrac{C_2}{C_1}+3.\dfrac{C_3}{C_2}+\cdots+n.\dfrac{C_n}{C_{n-1}}=\dfrac{n(n+1)}{2}$

Solution

(i) We know that

$$(1+x)^n = C_0+C_1.x+C_2.x^2+\cdots+C_n.x^n.$$

Put $x=3$, we get

$$4^n=(1+3)^n=C_0+C_1.3+C_2.3^2+\cdots+C_n.3^n.$$

(ii) $$\frac{C_1}{C_0}+2.\frac{C_2}{C_1}+3.\frac{C_3}{C_2}+\cdots+n\frac{C_n}{C_{n-1}}=\sum_{r=1}^{n} r.\frac{C_r}{C_{r-1}}$$

$$=\sum_{r=1}^{n} r.\frac{{}^nC_r}{{}^nC_{(r-1)}}=\sum_{r=1}^{n} r.\frac{n!}{(n-r)!r!}\times\frac{(n-r+1)!(r-1)!}{n!}$$

$$=\sum_{r=1}^{n}(n-r+1)=n+(n-1)+(n-2)+\ldots+1=\frac{n(n+1)}{2}.$$

12. Problem : *For* $r=0,1,2,\ldots,n$, *prove that*

$$C_0.C_r+C_1.C_{r+1}+C_2.C_{r+2}+\cdots+C_{n-r}.C_n={}^{2n}C_{(n+r)}$$

and hence deduce that

(i) $C_0^2+C_1^2+C_2^2+\cdots+C_n^2={}^{2n}C_n$

(ii) $C_0 . C_1 + C_1 . C_2 + C_2 . C_3 + \cdots + C_{n-1} . C_n = {}^{2n}C_{n+1}$

Solution : We know that

$$(1+x)^n = C_0 + C_1 . x + C_2 . x^2 + \cdots + C_n . x^n \qquad \text{... (1)}$$

On replacing 'x' by $\frac{1}{x}$, we get

$$\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \cdots + \frac{C_n}{x^n}. \qquad \text{... (2)}$$

On multiplying (1) and (2), we get

$$\left(1+\frac{1}{x}\right)^n (1+x)^n = \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \cdots + \frac{C_n}{x^n}\right)$$

$$(C_0 + C_1 . x + C_2 . x^2 + \cdots + C_n . x^n) \qquad \text{... (3)}$$

For $0 \le r \le n$, the coefficient of x^r in the expansion in R.H.S. of (3)

$$= C_0 . C_r + C_1 . C_{r+1} + C_2 . C_{r+2} + \cdots + C_{n-r} . C_n \qquad \text{... (4)}$$

Now, the coefficient of x^r in L.H.S of (3)

$= $ the coefficient of x^r in $\frac{(1+x)^{2n}}{x^n}$

$= $ the coefficient of x^{n+r} in $(1+x)^{2n}$

$$= {}^{2n}C_{n+r} \qquad \text{... (5)}$$

Hence, from (4) and (5), we get

$$C_0 . C_r + C_1 . C_{r+1} + C_2 . C_{r+2} + \ldots + C_{n-r} . C_n = {}^{2n}C_{n+r} \qquad \text{... (6)}$$

(i) on substituting $r = 0$ in (6), we get

$$C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = {}^{2n}C_n$$

(ii) on substituting $r = 1$ in (6), we get

$$C_0 . C_1 + C_1 . C_2 + C_2 . C_3 + \cdots + C_{n-1} . C_n = {}^{2n}C_{n+1}$$

13. Problem : *Prove that*

$$3\,.\,C_0{}^2+7\,.\,C_1{}^2+11\,.C_2{}^2+\cdots+(4n+3).C_n{}^2=(2n+3)\,.\,{}^{2n}C_n\,.$$

Solution : Write

$$S=3\,.\,C_0{}^2+7\,.\,C_1{}^2+11.\,C_2{}^2+\cdots+(4n+3)\,.\,C_n{}^2 \qquad \ldots (1)$$

On writing the terms of the R.H.S. of (1), in the reverse order, we get

$$\begin{aligned} S &= (4n+3)\,C_n{}^2+(4n-1)\,C^2{}_{n-1}+(4n-5)\,C^2{}_{n-2}+\cdots+3.\,C_0{}^2 \\ &= (4n+3)\,C_0{}^2+(4n-1)\,C_1{}^2+(4n-5)\,C_2{}^2+\ldots \\ &\qquad +3.C_n{}^2\ (\text{since } C_{n-r}=C_r,\ \text{for } 0\le r\le n) \end{aligned}$$

on adding (1) and (2) , we get

$$\begin{aligned} 2S &= (4n+6)\,C_0{}^2+(4n+6)\,C_1{}^2+(4n+6)\,C_2{}^2+\ldots+(4n+6).C_n{}^2 \\ &= (4n+6)\,(C_0{}^2+C_1{}^2+C_2{}^2+\ldots+C_n{}^2) \\ &= (4n+6)\,.\,{}^{2n}C_n \qquad \text{(from problem 10 (i) above)} \end{aligned}$$

$$\therefore\ S=(2n+3)\,.\,{}^{2n}C_n$$

14. Problem : Find the numerically greatest term(s) in the expansion of

(i) $(2+3x)^{10}$ when $x=\dfrac{11}{8}$ (ii) $(3x-4y)^{14}$ when $x=8,\ y=3$

Solution

(i) Write $(2+3x)^{10}=2^{10}\,(1+\dfrac{3x}{2})^{10}$ ... (1)

First we find the numerically greatest term in the expansion of $\left(1+\dfrac{3x}{2}\right)^{10}$.

We write $X=\dfrac{3x}{2}$ and calculate $\dfrac{(n+1)|X|}{1+|X|}$. Here $|X|=\left|\dfrac{3x}{2}\right|=\left|\dfrac{33}{16}\right|=\dfrac{33}{16}$. Now,

$$\frac{(n+1)|X|}{1+|X|}=\frac{11.\dfrac{33}{16}}{1+\dfrac{33}{16}}=\frac{363}{49}\ \text{ is not an integer.}$$

Its integral part $m=\left[\dfrac{(n+1)|X|}{1+|X|}\right]=\left[\dfrac{363}{49}\right]=7$.

Hence, by Theorem 6.1.18, T_{m+1} is the numerically greatest term in the binomial expansion of $(1+x)^n = \left(1+\frac{3x}{2}\right)^{10}$ and

$$T_{m+1} = T_8 = {}^{10}C_7 \cdot \left(\frac{3x}{2}\right)^7 = {}^{10}C_7 \cdot \left(\frac{33}{16}\right)^7.$$

Therefore, from (1) the numerically greatest term in the expansion of $(2+3x)^{10}$

$$= 2^{10} \cdot {}^{10}C_7 \cdot \left(\frac{33}{16}\right)^7$$

(ii) we have $(3x-4y)^{14} = (3x)^{14}\left(1-\frac{4y}{3x}\right)^{14}$ ... (2)

First we find the numerically greatest term in the expansion of $\left(1-\frac{4y}{3x}\right)^{14}$

write $X = -\frac{4y}{3x}$ then $|X| = \left|-\frac{4y}{3x}\right| = \frac{4}{3} \cdot \frac{3}{8} = \frac{1}{2}$. Now

$$m = \frac{(n+1)|X|}{1+|X|} = \frac{15 \cdot \frac{1}{2}}{1+\frac{1}{2}} = 5, \text{ an integer.}$$

Hence, by Theorem 6.1.18, $|T_m| = |T_{m+1}|$ and

T_m, T_{m+1} are both numerically greatest terms in the expansion of $\left(1-\frac{4y}{3x}\right)^{14}$.

Now $$T_m = T_5 = {}^{14}C_4 \cdot \left(\frac{-4y}{3x}\right)^4 = {}^{14}C_4 \cdot \left(\frac{1}{2}\right)^4$$

and $$T_{m+1} = T_6 = {}^{14}C_5 \cdot \left(\frac{-4y}{3x}\right)^5 = -{}^{14}C_5 \cdot \left(\frac{1}{2}\right)^5.$$

Therefore the numerically greatest terms in the expansion of $(3x-4y)^{14}$ are T_5 and T_6. They are

$$T_5 = (24)^{14} \cdot {}^{14}C_4 \left(\frac{1}{2}\right)^4 \quad \text{and} \quad T_6 = -(24)^{14} \cdot {}^{14}C_5 \left(\frac{1}{2}\right)^5.$$

15. Problem : *Prove that* $6^{2n} - 35n - 1$ *is divisible by 1225 for all natural numbers* n.

Solution : $6^{2n} = (36)^n = (35+1)^n$

$$= 35^n + {}^nC_1 . 35^{n-1} + \cdots + {}^nC_{n-2} . 35^2 + {}^nC_{n-1} . 35 + 1$$

$$= 35^2 \left(35^{n-2} + {}^nC_1 \; 35^{n-3} + \cdots + {}^nC_{n-2}\right) + 35n + 1 \text{ if } n \geq 2$$

i.e., $6^{2n} - 35n - 1 = 1225(k)$ for some integer k (if $n \geq 2$)

If $n = 1$, then $6^{2n} - 35n - 1 = 6^2 - 35 - 1 = 0$, which is trivially divisible by 1225.

Hence, for all natural numbers n, $6^{2n} - 35n - 1$ is divisible by 1225.

Note : The above problem can also be proved by induction.

16. Problem : *Suppose that* n *is a natural number and I, F are respectively the integral part and fractional part of* $(7+4\sqrt{3})^n$. *Then show that*

(i) I is an odd integer (ii) $(I+F)(1-F) = 1$.

Solution : Given that $(7+4\sqrt{3})^n = I + F$ where I is an integer and $0 < F < 1$.

Write $f = (7-4\sqrt{3})^n$. Now

from $36 < 48 < 49$, we get that

$$6 < \sqrt{48} < 7$$

i.e., $-7 < -\sqrt{48} < -6$

i.e., $0 < 7 - 4\sqrt{3} < 1$

i.e., $0 < (7-4\sqrt{3})^n < 1$

Therefore, $0 < f < 1$

Now,

$$I + F + f = (7+4\sqrt{3})^n + (7-4\sqrt{3})^n$$

$$= \left({}^nC_0 \, 7^n + {}^nC_1 . 7^{n-1} (4\sqrt{3})^1 + {}^nC_2 . 7^{n-2} (4\sqrt{3})^2 + \cdots\right)$$
$$+ \left({}^nC_0 . 7^n - {}^nC_1 . 7^{n-1} (4\sqrt{3})^1 + {}^nC_2 7^{n-2} (4\sqrt{3})^2 \ldots\right)$$

$$= 2\left(7^n + {}^nC_2 . 7^{n-2} . (4\sqrt{3})^2 + {}^nC_4 \, 7^{n-4} . (4\sqrt{3})^4 + \cdots\right)$$

$= 2k$ where k is a positive integer ... (1)

Thus, $I+F+f$ is an even integer. Since I is an integer, we get that

$F+f$ is an integer. Also, since $0<F<1$ and $0<f<1$,

we get $0<F+f<2$. Since $F+f$ is an integer, we get $F+f=1$ i.e $1-F=f$ ...(2)

(i) From (1), $I+F+f = 2k$

$$\Rightarrow I+1=2k \Rightarrow I = 2k-1, \text{ an odd integer}$$

(ii) $(I+F)(1-F) = (I+F)f$ from (2)

$$= (7+4\sqrt{3})^n\,(7-4\sqrt{3})^n = (49-48)^n = 1.$$

17. Problem : *Find the coefficient of* x^6 *in* $(3+2x+x^2)^6$

Solution : $(3+2x+x^2)^6 = ((3+2x)+x^2)^6$

$$= {}^6C_0\,(3+2x)^6.(x^2)^0 + {}^6C_1(3+2x)^5.(x^2)^1 + {}^6C_2(3+2x)^4.(x^2)^2 + {}^6C_3(3+2x)^3.(x^2)^3+\cdots$$

$$= (3+2x)^6 + 6x^2(3+2x)^5 + 15x^4(3+2x)^4 + 20x^6(3+2x)^3 +\cdots$$

$$= \left(\sum_{r=0}^{6} {}^6C_r.3^{6-r}.(2x)^r\right) + 6x^2\left(\sum_{r=0}^{5} {}^5C_r.3^{5-r}.(2x)^r\right) + 15x^4\left(\sum_{r=0}^{4} {}^4C_r\,3^{4-r}.(2x)^r\right) + 20x^6\left(\sum_{r=0}^{3} {}^3C_r\,3^{3-r}.(2x)^r\right)+\cdots$$

Therefore, the coefficient of x^6 in the expansion of $(3+2x+x^2)^6$

$$= ({}^6C_6.3^0.2^6) + 6({}^5C_4.3^1.2^4) + 15({}^4C_2.3^2.2^2) + 20({}^3C_0.3^3.2^0)$$
$$= 64 + 1440 + 3240 + 540 = 5284.$$

ALITER : From 6.1.6, we have

$$(a+b+c)^n = \sum_{\substack{0\le p,q,r\le n\\ p+q+r=n}} \frac{n!}{p!q!r!}\,a^p.b^q.c^r$$

$$\left(3+2x+x^2\right)^6 = \sum_{\substack{0\le p,q,r\le 6\\ p+q+r=6}} \frac{6!}{p!q!r!}\,3^p(2x)^q.(x^2)^r$$

$$= \sum_{\substack{0\le p,q,r\le 6\\ p+q+r=6}} \frac{6!}{p!q!r!}\,3^p.2^q.x^{q+2r}$$

To get the coefficient of x^6 , we take all possible values of p, q, r

such that $0 \le p, q, r \le 6$, $p + q + r = 6$ and $q + 2r = 6$. Put

(i) $q = 0,\ r = 3,\ p = 3$: The coefficient of x^6 is $\dfrac{6!}{3!\,0!\,3!}\,3^3 . 2^0 = (20)(27) = 540$.

(ii) $q = 2,\ r = 2,\ p = 2$: The coefficient of x^6 is $\dfrac{6!}{2!\,2!\,2!}\,3^2 . 2^2 = 90 \times 36 = 3240$.

(iii) $q = 4,\ r = 1,\ p = 1$: The coefficient of x^6 is $\dfrac{6!}{4!\,1!\,1!}\,3^1 . 2^4 = 30 \times 48 = 1440$.

(iv) $q = 6,\ r = 0,\ p = 0$: The coefficient of x^6 is $\dfrac{6!}{6!\,0!\,0!} . 3^0 . 2^6 = 64$.

Thus the coefficient of x^6 in the expansion of $(3 + 2x + x^2)^6$

$$= 540 + 3240 + 1440 + 64 = 5284.$$

18. Problem : *If n is a positive integer, then prove that*

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}.$$

Solution : Write $S = C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \cdots + \dfrac{C_n}{n+1}$.

$$\text{Then } S = {}^nC_0 + \frac{1}{2} . {}^nC_1 + \frac{1}{3} . {}^nC_2 + \cdots + \frac{1}{n+1} . {}^nC_n$$

$$\text{Therefore, } (n+1)\,S = \frac{n+1}{1}\,{}^nC_0 + \frac{n+1}{2} . {}^nC_1 + \frac{n+1}{3} . {}^nC_2 + \cdots + \frac{n+1}{n+1} . {}^nC_n$$

$$= {}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \cdots + {}^{n+1}C_{n+1}$$

$$\left(\text{since } \frac{n+1}{r+1} . {}^nC_r = {}^{(n+1)}C_{(r+1)}\right)$$

$$= 2^{n+1} - {}^{(n+1)}C_0 = 2^{n+1} - 1. \quad \text{Hence,} \quad S = \frac{2^{n+1} - 1}{n+1}.$$

The next problem is a generalization of the above one.

19. Problem : *If n is a positive integer and x is any nonzero real number, then prove that*

$$C_0 + C_1 . \frac{x}{2} + C_2 . \frac{x^2}{3} + C_3 . \frac{x^3}{4} + \cdots + C_n . \frac{x^n}{n+1} = \frac{(1+x)^{n+1} - 1}{(n+1)x}$$

Solution : Write $S = C_0 + C_1 . \frac{x}{2} + C_2 . \frac{x^2}{3} + \cdots + C_n . \frac{x^n}{n+1}$

$$\Rightarrow x.S = C_0 . x + C_1 . \frac{x^2}{2} + C_2 . \frac{x^3}{3} + \cdots + C_n . \frac{x^{n+1}}{n+1}$$

$$\Rightarrow (n+1)\ x.S = \frac{n+1}{1} . {}^nC_0 . x + \frac{n+1}{2} . {}^nC_1 . x^2 + \frac{n+1}{3} . {}^nC_2 . x^3 + \cdots + \frac{n+1}{n+1} . {}^nC_n . x^{n+1}$$

$$= {}^{(n+1)}C_1 . x + {}^{(n+1)}C_2 . x^2 + {}^{(n+1)}C_3 . x^3 + \cdots + {}^{(n+1)}C_{(n+1)} . x^{n+1}$$

$$= (1+x)^{n+1} - {}^{(n+1)}C_0 = (1+x)^{n+1} - 1.$$

Therefore, $S = \dfrac{(1+x)^{n+1} - 1}{(n+1)\ x}$.

20. Problem : *Prove that*

$$C_0^2 - C_1^2 + C_2^2 - C_3^2 + \cdots + (-1)^n . C_n^2 = \begin{cases} (-1)^{\frac{n}{2}} . {}^nC_{\frac{n}{2}} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Solution : Consider

$$(1-x)^n \left(1 + \frac{1}{x}\right)^n = (C_0 - C_1 x + C_2 x^2 - C_3 x^3 + \cdots + (-1)^n . C_n x^n)$$
$$\times \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{C_3}{x^3} + \cdots + \frac{C_n}{x^n}\right) \qquad \ldots (1)$$

The term independent to x (that is, the constant term) in R.H.S. of (1)

$$= C_0^2 - C_1^2 + C_2^2 - C_3^2 + \cdots + (-1)^n . C_n^2$$

Now, we calculate the term independent of x in the L.H.S. of equation (1). From (1)

$$\text{L.H.S.} = (1-x)^n \left(1 + \frac{1}{x}\right)^n = (1-x)^n \frac{(1+x)^n}{x^n} = \frac{(1-x^2)^n}{x^n} = \frac{\sum_{r=o}^{n} {}^nC_r (-x^2)^r}{x^n} \qquad \ldots (2)$$

Observe that the expansion in the numerator of (2) contains only even powers of x. Therefore, if n is odd, then there is no constant term in (2). In other words, the term independent of x in $(1-x)^n (1 + \frac{1}{x})^n$ is zero.

Now, suppose n is an even integer say $n = 2k$. Then, from (2) we get

$$(1-x)^n\,(1+\frac{1}{x})^n = \frac{\sum\limits_{r=o}^{n} {}^{n}C_r(-x^2)^r}{x^n} = \frac{\sum\limits_{r=o}^{2k} {}^{2k}C_r(-x^2)^r}{x^{2k}} = \sum_{r=o}^{2k} {}^{2k}C_r(-1)^r\,.\,x^{2r-2k} \quad \ldots (3)$$

To get the term independent of x in (3), put $2r - 2k = 0$, Then $r = k$ and hence the term independent of x in $(1-x)^n\left(1+\frac{1}{x}\right)^n$ is ${}^{2k}C_k\,.\,(-1)^k = {}^{n}C_{\frac{n}{2}}\,(-1)^{\frac{n}{2}}$

Hence, from (1), we get

$$C_0^{\,2} - C_1^{\,2} + C_2^{\,2} - C_3^{\,2} + \cdots + (-1)^n\,.\,C_n^{\,2} = \begin{cases} (-1)^{\frac{n}{2}}\,.\,{}^{n}C_{\frac{n}{2}}, & \text{if } n \text{ is even} \\ 0 & \text{, if } n \text{ is odd} \end{cases}$$

Exercise 6(a)

I. 1. Expand the following using binomial theorem

(i) $(4x+5y)^7$ (ii) $\left(\frac{2}{3}x+\frac{7}{4}y\right)^5$ (iii) $\left(\frac{2p}{5}-\frac{3q}{7}\right)^6$ (iv) $(3+x-x^2)^4$

2. Write down and simplify

(i) 6^{th} term in $\left(\frac{2x}{3}+\frac{3y}{2}\right)^9$ (ii) 7^{th} term in $(3x-4y)^{10}$

(iii) 10^{th} term in $\left(\frac{3p}{4}-5q\right)^{14}$ (iv) r^{th} term in $\left(\frac{3a}{5}+\frac{5b}{7}\right)^8$ $(1 \le r \le 9)$.

3. Find the number of terms in the expansion of

(i) $\left(\frac{3a}{4}+\frac{b}{2}\right)^9$ (ii) $(3p+4q)^{14}$ (iii) $(2x+3y+z)^7$

4. Find the number of terms with non-zero coefficients in $(4x-7y)^{49}+(4x+7y)^{49}$.

5. Find the sum of last 20 coefficients in the expansion of $(1+x)^{39}$.

6. If A and B are coefficients of x^n in the expansion of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, then find the value of $\frac{A}{B}$.

II. 1. Find the coefficient of

(i) x^{-6} in $\left(3x-\dfrac{4}{x}\right)^{10}$ (ii) x^{11} in $\left(2x^2+\dfrac{3}{x^3}\right)^{13}$

(iii) x^2 in $\left(7x^3-\dfrac{2}{x^2}\right)^9$ (iv) x^{-7} in $\left(\dfrac{2x^2}{3}-\dfrac{5}{4x^5}\right)^7$

2. Find the term independent of x in the expansion of

(i) $\left(\dfrac{\sqrt{x}}{3}-\dfrac{4}{x^2}\right)^{10}$ (ii) $\left(\dfrac{3}{\sqrt[3]{x}}+5\sqrt{x}\right)^{25}$ (iii) $\left(4x^3+\dfrac{7}{x^2}\right)^{14}$ (iv) $\left(\dfrac{2x^2}{5}+\dfrac{15}{4x}\right)^9$

3. Find the middle term(s) in the expansion of

(i) $\left(\dfrac{3x}{7}-2y\right)^{10}$ (ii) $\left(4a+\dfrac{3}{2}b\right)^{11}$ (iii) $\left(4x^2+5x^3\right)^{17}$ (iv) $\left(\dfrac{3}{a^3}+5a^4\right)^{20}$

4. Find the numerically greatest term(s) in the expansion of

(i) $(4+3x)^{15}$ when $x=\dfrac{7}{2}$ (ii) $(3x+5y)^{12}$ when $x=\dfrac{1}{2},\ y=\dfrac{4}{3}$

(iii) $(4a-6b)^{13}$ when $a=3,\ b=5$ (iv) $(3+7x)^n$ when $x=\dfrac{4}{5},\ n=15$

5. Prove the following :

(i) $2.C_o+5.C_1+8.C_2+\cdots+(3n+2)C_n=(3n+4)2^{n-1}$

(ii) $C_o-4.C_1+7.C_2-10.C_3+\cdots=0$, if n is an even positive integer.

(iii) $\dfrac{C_1}{2}+\dfrac{C_3}{4}+\dfrac{C_5}{6}+\dfrac{C_7}{8}+\cdots=\dfrac{2^n-1}{n+1}$

(iv) $C_o+\dfrac{3}{2}.C_1+\dfrac{9}{3}.C_2+\dfrac{27}{4}.C_3+\cdots+\dfrac{3^n}{n+1}C_n=\dfrac{4^{n+1}-1}{3(n+1)}$

(v) $C_o+2.C_1+4.C_2+8.C_3+\cdots+2^n.C_n=3^n$

6. Find the sum of the following

(i) $\dfrac{^{15}C_1}{^{15}C_0}+2\dfrac{^{15}C_2}{^{15}C_1}+3\dfrac{^{15}C_3}{^{15}C_2}+\cdots+15\dfrac{^{15}C_{15}}{^{15}C_{14}}$

(ii) $C_0.C_3+C_1.C_4+C_2.C_5+\cdots+C_{n-3}.C_n$

(iii) $2^2C_0 + 3^2C_1 + 4^2C_2 + ... + (n+2)^2C_n$

(iv) $3C_0 + 6C_1 + 12C_2 + ... + 3.2^nC_n$

7. Using binomial theorem prove that $50^n - 49n - 1$ is divisible by 49^2 for all positive integers *n*.

8. Using binomial theorem, prove that $5^{4n} + 52n - 1$ is divisible by 676 for all positive integers *n*.

9. If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, then prove that

(i) $a_0 + a_1 + a_2 + \cdots + a_{2n} = 3^n$ (ii) $a_0 + a_2 + a_4 + \cdots + a_{2n} = \dfrac{3^n+1}{2}$

(iii) $a_1 + a_3 + a_5 + \cdots + a_{2n-1} = \dfrac{3^n-1}{2}$ (iv) $a_0 + a_3 + a_6 + a_9 + \cdots = 3^{n-1}$

10. If $(1+x+x^2+x^3)^7 = b_0 + b_1x + b_2x^2 + ... + b_{21}x^{21}$, then find the value of

(i) $b_0 + b_2 + b_4 + ... + b_{20}$ (ii) $b_1 + b_3 + b_5 + ... + b_{21}$

11. If the coefficients of x^{11} and x^{12} in the binomial expansion of $\left(2 + \dfrac{8x}{3}\right)^n$ are equal, find *n*.

12. Find the remainder when 2^{2013} is divided by 17.

13. If the coefficents of $(2r+4)^{th}$ term and $(3r+4)^{th}$ term in the expansion of $(1+x)^{21}$ are equal, find r.

III. 1. If the coefficients of x^9, x^{10}, x^{11} in the expansion of $(1+x)^n$ are in A.P. then prove that $n^2 - 41n + 398 = 0$.

2. If 36, 84, 126 are three successive binomial coefficients in the expansion of $(1+x)^n$, then find *n*.

3. If the 2nd, 3rd and 4th terms in the expansion of $(a+x)^n$ are respectively 240, 720, 1080, find *a, x, n*.

4. If the coefficients of r^{th}, $(r+1)^{th}$, and $(r+2)^{nd}$ terms in the expansion of $(1+x)^n$ are in A.P., then show that

$$n^2 - (4r+1)n + 4r^2 - 2 = 0.$$

5. Find the sum of the coefficients of x^{32} and x^{-18} in the expansion of $\left(2x^3 - \dfrac{3}{x^2}\right)^{14}$.

6. If P and Q are the sum of odd terms and the sum of even terms respectively in the expansion of $(x+a)^n$ then prove that

   (i) $P^2 - Q^2 = (x^2 - a^2)^n$ (ii) $4PQ = (x+a)^{2n} - (x-a)^{2n}$

7. If the coefficients of 4 consecutive terms in the expansion of $(1+x)^n$ are a_1, a_2, a_3, a_4 respectively, then show that

$$\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}.$$

8. Prove that $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - ({}^{2n}C_3)^2 + \cdots + ({}^{2n}C_{2n})^2 = (-1)^n . {}^{2n}C_n$.

9. Prove that

$$(C_0+C_1)(C_1+C_2)(C_2+C_3)\ldots(C_{n-1}+C_n) = \frac{(n+1)^n}{n!}. C_0 . C_1 . C_2 \ldots\ldots\ldots C_n.$$

10. Find the term independent of x in $(1+3x)^n\left(1+\frac{1}{3x}\right)^n$.

11. Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5.....(2n-1)}{n!}(2x)^n$.

12. If $\left(1+3x-2x^2\right)^{10} = a_0 + a_1x + a_2x^2 + \ldots + a_{20}x^{20}$, then prove that

    (i) $a_0 + a_1 + a_2 + \ldots + a_{20} = 2^{10}$ (ii) $a_0 - a_1 + a_2 - a_3 + \ldots + a_{20} = 4^{10}$

13. If $(3\sqrt{3}+5)^{2n+1} = x$ and $f = x - [x]$ (where $[x]$ is the integral part of x), find the value of $x.f$.

14. If R, n are positive integers, n is odd, $0 < F < 1$ and if $\left(5\sqrt{5}+11\right)^n = R + F$, then prove that

    (i) R is an even integer and (ii) $(R + F) . F = 4^n$

15. If I, n are positive integers, $0 < f < 1$ and if $\left(7+4\sqrt{3}\right)^n = I + f$, then show that

    (i) I is an odd integer and (ii) $(I + f)(1 - f) = 1$

16. If n is a positive integer, prove that $\sum_{r=1}^{n} r^3 . \left(\frac{{}^nC_r}{{}^nC_{r-1}}\right)^2 = \frac{n(n+1)^2(n+2)}{12}$.

17. Find the number of irrational terms in the expansion of $(5^{1/6} + 2^{1/8})^{100}$.

6.2 Binomial Theorem for Rational Index

In the previous section, we have proved that

$$(1+x)^n = {}^nC_0 + {}^nC_1\ x + {}^nC_2\ x^2 + \ldots + {}^nC_r\ x^r + \ldots + {}^nC_n\ x^n$$

where n is a positive integer and x is any real number. We can write this also in the form

$$(1+x)^n = 1 + \frac{n}{1}\ x + \frac{n(n-1)}{2!}\ x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}\ x^r + \ldots \qquad \ldots (1)$$

The R.H.S. of the above equality terminates automatically after $(n+1)$ terms, when n is a positive integer since we come across the factor $(n-n)$ from the $(n+2)^{nd}$ term onwards. But in case n is a negative integer or, more generally, a rational number $\left(\frac{p}{q}\right)$ which is not a positive integer, then (1) contains infinite number of terms in R.H.S. Still it becomes valid provided we stipulate the condition $|x| < 1$. (The proof of this fact is beyond the scope of this book).

Now, we state without proof, the binomial theorem for rational index.

6.2.1 Theorem (Binomial Theorem for Rational Index)

If m is a rational number (but not a positive integer) and x is a real number such that $|x|<1$ (that is, $-1<x<1$), then

$$(1+x)^m = 1 + \frac{m}{1!}\ x + \frac{m(m-1)}{2!}\ x^2 + \ldots + \frac{m(m-1)\ldots(m-r+1)}{r!}\ x^r + \ldots$$

$$= 1 + \sum_{r=1}^{\infty} \frac{m(m-1)\ldots(m-r+1)}{r!}\ x^r \qquad \ldots (*)$$

Now, we discuss some special cases.

Negative Integral Index

Let m be a negative integer, say $m=-n$ (n is a positive integer) and $|x|<1$. Then

1. $$(1+x)^{-n} = 1 + \frac{(-n)}{1!}x + \frac{(-n)(-n-1)}{2!}\ x^2 + \cdots + \frac{(-n)(-n-1)\ldots(-n-r+1)}{r!}x^r + \cdots$$

$$= 1 - \frac{n}{1!}\ x + \frac{n(n+1)}{2!}\ x^2 - \cdots + (-1)^r\ \frac{n(n+1)\ldots(n+r-1)}{r!}\ x^r + \cdots$$

$$\therefore\ (1+x)^{-n} = \sum_{r=0}^{\infty} (-1)^r\ {}^{n+r-1}C_r\,.x^r$$

2. On replacing x by $-x$ in the above, we get

$$(1-x)^{-n} = 1 - \frac{n}{1!}(-x) + \frac{n(n+1)}{2!}(-x)^2 - \frac{n(n+1)(n+2)}{3!}(-x)^3 + \cdots$$

$$+ (-1)^r\,.\ \frac{n(n+1)\ldots(n+r-1)}{r!}(-x)^r + \cdots$$

$$= 1 + \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$$

$$+ \frac{n(n+1)\ldots(n+r-1)}{r!}x^r + \cdots$$

$$\therefore\ (1-x)^{-n} = \sum_{r=0}^{\infty} {}^{(n+r-1)}C_r\,.\,x^r$$

Rational (non-integer) Index

Let m be a positive rational number, say $m = \frac{p}{q}$ where p, q are positive integers ($q \neq 1$) and x a real number such that $|x| < 1$.

3. $$(1+x)^{\frac{p}{q}} = 1 + \frac{\frac{p}{q}}{1!}x + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)}{2!}x^2 + \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-2\right)}{3!}x^3 + \cdots$$

$$+ \frac{\frac{p}{q}\left(\frac{p}{q}-1\right)\ldots\left(\frac{p}{q}-r+1\right)}{r!}x^r + \cdots$$

$$= 1 + \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p-q)}{2!}\left(\frac{x}{q}\right)^2 + \frac{p(p-q)(p-2q)}{3!}\left(\frac{x}{q}\right)^3 + \cdots$$

$$+ \frac{p(p-q)\ \ldots\ (p-(r-1)q)}{r!}\left(\frac{x}{q}\right)^r + \cdots$$

Similarly, we can derive that

4. $$(1-x)^{\frac{p}{q}} = 1-\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p-q)}{2!}\left(\frac{x}{q}\right)^2-\frac{p(p-q)(p-2q)}{3!}\left(\frac{x}{q}\right)^3+\cdots$$
$$+(-1)^r\cdot\frac{p(p-q)\ldots(p-(r-1)q)}{r!}\left(\frac{x}{q}\right)^r+\cdots$$

5. $$(1+x)^{\frac{-p}{q}} = 1-\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2-\frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3+\cdots$$
$$+(-1)^r\cdot\frac{p(p+q)\ldots(p+(r-1)q)}{r!}\left(\frac{x}{q}\right)^r+\cdots$$

6. $$(1-x)^{\frac{-p}{q}} = 1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+\frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3+\ldots$$
$$+\frac{p(p+q)\ldots(p+(r-1)q)}{r!}\left(\frac{x}{q}\right)^r+\cdots$$

6.2.2 Note

We observe the following important facts in the above expansions for rational index, which will help us to distinguish one from the other.

(i) The terms in the expansion of $(1-x)^{\frac{p}{q}}$ and $(1+x)^{\frac{-p}{q}}$ have alternately positive and negative signs while the terms in the other two expansions $(1+x)^{\frac{p}{q}}$ and $(1-x)^{\frac{-p}{q}}$ have all positive signs.

(ii) In the numerators of the coefficients of the terms in the expansions of $(1+x)^{\frac{p}{q}}$ or $(1-x)^{\frac{p}{q}}$, p is followed by the terms $(p-q), (p-2q)$ etc. and while the numerators of the coefficients of the terms in the expansions of $(1+x)^{\frac{-p}{q}}$ or $(1-x)^{\frac{-p}{q}}$, p is followed by $(p+q), (p+2q)$ etc.

(iii) In general, we can use the expansion (*) for all the 6 cases described above. That is, if p, q are integers with $q \neq 0$ and x is real number such that $|x|<1$, then

$$(1+x)^{\frac{p}{q}} = 1+\frac{\frac{p}{q}}{1!}x+\frac{\frac{p}{q}(\frac{p}{q}-1)}{2!}x^2+\frac{\frac{p}{q}(\frac{p}{q}-1)(\frac{p}{q}-2)}{3!}\cdot x^3+\cdots$$
$$+\frac{\frac{p}{q}(\frac{p}{q}-1)\ldots(\frac{p}{q}-r+1)}{r!}x^r+\cdots$$

(for example, if $p=-n,\ q=1,$ then we get the expansion (1))

The general term (i.e., $(r+1)^{th}$ term) in the above expansion is

$$T_{r+1} = \frac{\frac{p}{q}(\frac{p}{q}-1)\ldots(\frac{p}{q}-(r-1))}{r!}.x^r, \text{ for } r \geq 1$$

In the following we give some particular cases of the expansions given above for ready use.

6.2.3 Note

If x is a real number and $|x| < 1$, then

(i) $(1+x)^{\frac{-1}{2}} = 1 - \frac{1}{1!}\left(\frac{x}{2}\right) + \frac{1.3}{2!}\left(\frac{x}{2}\right)^2 - \frac{1.3.5}{3!}\left(\frac{x}{2}\right)^3 + \ldots + (-1)^r.\frac{1.3.5\ldots(2r-1)}{r!}\left(\frac{x}{2}\right)^r + \cdots \infty$

(ii) $(1+x)^{\frac{-3}{2}} = 1 - \frac{3}{1!}\left(\frac{x}{2}\right) + \frac{3.5}{2!}\left(\frac{x}{2}\right)^2 - \frac{3.5.7}{3!}\left(\frac{x}{2}\right)^3 + \ldots + (-1)^r\frac{3.5.7\ldots(2r+1)}{r!}\left(\frac{x}{2}\right)^r + \cdots \infty$

(iii) $(1+x)^{-3} = 1 - 3.x + \frac{3.4}{1.2}x^2 - \frac{4.5}{1.2}x^3 + \cdots + (-1)^r . \frac{(r+1)\ (r+2)}{1.2}x^r + \cdots \infty$

On replacing x by $-x$ in the three expansions above, we get

(iv) $(1-x)^{\frac{-1}{2}} = 1 + \frac{1}{1!}\left(\frac{x}{2}\right) + \frac{1.3}{2!}\left(\frac{x}{2}\right)^2 + \frac{1.3.5}{3!}\left(\frac{x}{2}\right)^3 + \ldots + .\frac{1.3.5\ldots(2r-1)}{r!}\left(\frac{x}{2}\right)^r + \cdots \infty$

(v) $(1-x)^{\frac{-3}{2}} = 1 + \frac{3}{1!}\left(\frac{x}{2}\right) + \frac{3.5}{2!}\left(\frac{x}{2}\right)^2 + \frac{3.5.7}{3!}\left(\frac{x}{2}\right)^3 + \ldots + \frac{3.5.7\ldots(2r+1)}{r!}\left(\frac{x}{2}\right)^r + \cdots \infty$

(vi) $(1-x)^{-3} = 1 + 3x + \frac{3.4}{1.2}x^2 + \frac{4.5}{1.2}x^3 + \ldots + \frac{(r+1)(r+2)}{1.2}x^r + \ldots \infty$

6.2.4 Solved Problems

1. Problem : *Find the set* E *of the values of x for which the binomial expansions for the following are valid*

(i) $(3-4x)^{\frac{3}{4}}$ (ii) $(2+5x)^{\frac{-1}{2}}$ (iii) $(7-4x)^{-5}$ (iv) $(4+9x)^{\frac{-2}{3}}$ (v) $(a+bx)^r$

Solution

(i) $(3-4x)^{\frac{3}{4}} = 3^{\frac{3}{4}}\left(1-\frac{4x}{3}\right)^{\frac{3}{4}}$

Hence, the binomial expansion of $(3-4x)^{\frac{3}{4}}$ is valid when $\left|\frac{4x}{3}\right|<1$, that is, $|x|<\frac{3}{4}$;

Therefore, E = $\left(-\frac{3}{4},\frac{3}{4}\right)$.

(ii) $(2+5x)^{\frac{-1}{2}} = 2^{\frac{-1}{2}}\left(1+\frac{5x}{2}\right)^{\frac{-1}{2}}$.

Therefore, the binomial expansion of $(2+5x)^{\frac{-1}{2}}$ is valid when $\left|\frac{5x}{2}\right|<1$. That is, $|x|<\frac{2}{5}$.

Hence E = $\left(\frac{-2}{5},\frac{2}{5}\right)$

(iii) $(7-4x)^{-5} = 7^{-5}\left(1-\frac{4x}{7}\right)^{-5}$.

Hence, the binomial expansion of $(7-4x)^{-5}$ is valid when

$\left|\frac{4x}{7}\right|<1$ or $|x|<\frac{7}{4}$. Thus, E = $\left(\frac{-7}{4},\frac{7}{4}\right)$.

(iv) $(4+9x)^{\frac{-2}{3}} = 4^{\frac{-2}{3}}\left(1+\frac{9x}{4}\right)^{\frac{-2}{3}}$

Therefore, the binomial expansion of $(4+9x)^{\frac{-2}{3}}$ is valid when

$\left|\frac{9x}{4}\right|<1$ or $|x|<\frac{4}{9}$. Thus, E = $\left(\frac{-4}{9},\frac{4}{9}\right)$.

(v) For any nonzero reals a and b, the set of x for which the binomial expansion of $(a\pm bx)^r$ is valid when $r\notin \mathbf{Z}^+\cup\{0\}$, is $\left(-\frac{|a|}{|b|},\frac{|a|}{|b|}\right)$.

2. Problem : *Find the*

(i) 9^{th} *term of* $\left(2+\frac{x}{3}\right)^{-5}$

(ii) 10^{th} *term of* $\left(1-\frac{3x}{4}\right)^{\frac{4}{5}}$

(iii) 8^{th} *term of* $\left(1-\frac{5x}{2}\right)^{\frac{-3}{5}}$

(iv) 6^{th} *term of* $\left(3+\frac{2x}{3}\right)^{\frac{3}{2}}$

Solution

(i) $\left(2+\frac{x}{3}\right)^{-5} = 2^{-5}\left(1+\frac{x}{6}\right)^{-5}$ ... (1)

Consider $\left(1+\frac{x}{6}\right)^{-5}$. Comparing this with $(1+X)^{-n}$ where $X = \frac{x}{6}$, $n=5$,

the general term in the binomial expansion of $(1+X)^{-n}$ is

$T_{r+1} = (-1)^r \, .^{(n+r-1)}C_r . X^r \quad (0 \le r \le n)$ (from 6.2.1).

Put $r=8$, we get $T_9 = (-1)^8 \, .^{12}C_8 . \left(\frac{x}{6}\right)^8 = {}^{12}C_8 . \left(\frac{x}{6}\right)^8$

Hence, from (1) we get that the 9^{th} term of $\left(2+\frac{x}{3}\right)^{-5}$

$$= 2^{-5} . \, {}^{12}C_8 . \left(\frac{x}{6}\right)^8 = \frac{495}{32}\left(\frac{x}{6}\right)^8 .$$

(ii) On comparing $\left(1-\frac{3x}{4}\right)^{\frac{4}{5}}$ with $(1-X)^{\frac{p}{q}}$, we get $X=\frac{3x}{4}$, $p=4$, $q=5$

The general term in this expansion is

$$T_{r+1} = (-1)^r . \frac{p(p-q)...(p-(r-1)q)}{r!}\left(\frac{X}{q}\right)^r$$

put $r=9$, we get $T_{10} = (-1)^9 . \frac{4(4-5)(4-10)...\,(4-40)}{9!}\left(\frac{3x}{20}\right)^9$

$$= -\frac{4.1.6.(11)....(36)}{9!}\left(\frac{3x}{20}\right)^9 .$$

(iii) On comparing $\left(1-\frac{5x}{2}\right)^{\frac{-3}{5}}$ with $(1-X)^{\frac{-p}{q}}$, we get $X=\frac{5x}{2}$, $p=3$, $q=5$

The general term in this expansion is

$$T_{r+1} = .\frac{p(p+q)...(p+(r-1)q)}{r!}\left(\frac{X}{q}\right)^r .$$

Put $r=7$, we get $T_8 = \frac{3(8)(13)...(33)}{7!}\left(\frac{x}{2}\right)^7$

(iv) $\left(3+\frac{2x}{3}\right)^{\frac{3}{2}} = 3^{\frac{3}{2}}\left(1+\frac{2x}{9}\right)^{\frac{3}{2}}$. .... (2)

Comparing $\left(1+\frac{2x}{9}\right)^{\frac{3}{2}}$ with $(1+X)^{\frac{p}{q}}$, we get $X = \frac{2x}{9}$, $p=3$, $q=2$

The general term in this expansion is

$$T_{r+1} = .\frac{p(p-q)\cdots(p-(r-1)q)}{r!}\left(\frac{X}{q}\right)^r$$

put $r=5$, we get $\quad T_6 = \dfrac{3.1.-1.-3-5}{5!}\left(\dfrac{x}{9}\right)^5 = -\dfrac{3}{8}.\left(\dfrac{x}{9}\right)^5$

Hence, from (2), we get that the sixth term in the expansion of $\left(3+\dfrac{2x}{3}\right)^{\frac{3}{2}}$

$$= 3\sqrt{3}\left(-\frac{3}{8}.\left(\frac{x}{9}\right)^5\right) = -\frac{9\sqrt{3}}{8}\left(\frac{x}{9}\right)^5.$$

3. Problem : *Write the first 3 terms in the expansion of*

(i) $\left(1+\dfrac{x}{2}\right)^{-5}$ (ii) $(3+4x)^{\frac{-2}{3}}$ (iii) $(4-5x)^{\frac{-1}{2}}$

Solution

(i) We have $(1+X)^{-n} = 1-nX + \dfrac{n(n+1)}{2!}X^2 - \cdots$

Therefore $\quad \left(1+\dfrac{x}{2}\right)^{-5} = 1-\dfrac{5x}{2}+\dfrac{5.6}{2!}\left(\dfrac{x}{2}\right)^2 - \cdots$

Hence, the first 3 terms in the expansion of $\left(1+\dfrac{x}{2}\right)^{-5}$ are $1, \dfrac{-5x}{2}, \dfrac{15}{4}x^2$.

(ii) $(3+4x)^{\frac{-2}{3}} = 3^{\frac{-2}{3}}\left(1+\dfrac{4x}{3}\right)^{\frac{-2}{3}}$ ... (1)

We know that $(1+X)^{\frac{-p}{q}} = 1-\dfrac{p}{1}.\dfrac{X}{q} + \dfrac{p(p+q)}{1.2}\left(\dfrac{X}{q}\right)^2 - \cdots$

So that $\quad \left(1+\dfrac{4x}{3}\right)^{\frac{-2}{3}} = 1-\dfrac{2}{1}.\dfrac{4x}{9} + \dfrac{2.5}{1.2}\left(\dfrac{4x}{9}\right)^2 - \cdots$

Therefore, from (1), $(3+4x)^{\frac{-2}{3}} = 3^{\frac{-2}{3}}\left(1-\dfrac{8}{9}x+\dfrac{80}{81}x^2 - \cdots\right)$

Hence, the first 3 terms in the expansion of $(3+4x)^{\frac{-2}{3}}$ are $\dfrac{1}{3^{\frac{2}{3}}}, \dfrac{-8x}{3^{\frac{8}{3}}}, \dfrac{80x^2}{3^{\frac{14}{3}}}$.

(iii) $(4-5x)^{\frac{-1}{2}} = 4^{\frac{-1}{2}}(1-\frac{5x}{4})^{\frac{-1}{2}}$ ... (2)

We have, $(1-X)^{\frac{-p}{q}} = 1+\frac{p}{1}\cdot\frac{X}{q}+\frac{p(p+q)}{1.2}\left(\frac{X}{q}\right)^2+\cdots$

So that $\left(1-\frac{5x}{4}\right)^{\frac{-1}{2}} = 1+\frac{1}{1}\cdot\frac{5x}{8}+\frac{1.3}{1.2}\left(\frac{5x}{8}\right)^2+\cdots$

Therefore, from (2), $(4-5x)^{\frac{-1}{2}} = \frac{1}{2}\cdot\left(1+\frac{5x}{8}+\frac{75}{128}x^2+\ldots\right)$

Thus, the first 3 terms in the expansion of $(4-5x)^{\frac{-1}{2}}$ are $\frac{1}{2}, \frac{5x}{16}, \frac{75x^2}{256}$.

4. Problem : *Write the general term in the expansion of*

(i) $\left(3+\frac{x}{2}\right)^{\frac{-2}{3}}$ (ii) $\left(2+\frac{3x}{4}\right)^{\frac{4}{5}}$

(iii) $(1-4x)^{-3}$ (iv) $(2-3x)^{\frac{-1}{3}}$

Solution

(i) $\left(3+\frac{x}{2}\right)^{\frac{-2}{3}} = 3^{\frac{-2}{3}}\left(1+\frac{x}{6}\right)^{\frac{-2}{3}} = \frac{1}{3^{2/3}}\left(1+\frac{x}{6}\right)^{\frac{-2}{3}} = \frac{1}{\sqrt[3]{9}}\left(1+\frac{x}{6}\right)^{\frac{-2}{3}}$

Thus, the general term of $\left(3+\frac{x}{2}\right)^{\frac{-2}{3}}$ is

$$T_{r+1} = \frac{1}{\sqrt[3]{9}}\cdot\left\{(-1)^r\frac{p(p+q)\,(p+2q)\ldots(p+(r-1)q)}{r!}\left(\frac{X}{q}\right)^r\right\}$$

Where $p=2,\ q=3$ and $X=\frac{x}{6}$

Therefore, $T_{r+1} = \frac{1}{\sqrt[3]{9}}\cdot\left\{(-1)^r\frac{2.5.8\ldots(3r-1)}{r!}\left(\frac{x}{18}\right)^r\right\}$.

(ii) $\left(2+\frac{3x}{4}\right)^{\frac{4}{5}} = 2^{\frac{4}{5}}\left(1+\frac{3x}{8}\right)^{\frac{4}{5}} = \sqrt[5]{16}\left(1+\frac{3x}{8}\right)^{\frac{4}{5}}$.

Therefore, the general term of $\left(1+\dfrac{3x}{8}\right)^{\frac{4}{5}}$ is

$$T_{r+1} = \frac{p(p-q)(p-2q)\cdots(p-(r-1)q)}{r!}\left(\frac{X}{q}\right)^r$$

where $p = 4,\ q = 5,$ and $X = \dfrac{3x}{8}$

$$\text{Thus } T_{r+1} = \frac{4(-1)(-6)...(-5r+9)}{r!}\left(\frac{3x}{40}\right)^r$$

$$= (-1)^{r-1}\frac{4.1.6...(5r-9)}{r!}\left(\frac{3x}{40}\right)^r$$

Therefore, the general term of $\left(2+\dfrac{3x}{4}\right)^{\frac{4}{5}}$ is

$$T_{r+1} = \sqrt[5]{16}\,.\left\{(-1)^r\frac{4.1.6...(5r-9)}{r!}\left(\frac{3x}{40}\right)^r\right\}.$$

(iii) $(1-4x)^{-3} = (1-X)^{-n}$ where $X = 4x$ and $n = 3$

Now, the general term of $(1-4x)^{-3}$ is

$$T_{r+1} = {}^{(n+r-1)}C_r\,.X^r$$

$$= {}^{(3+r-1)}C_r\,.(4x)^r = {}^{(r+2)}C_2\,.(4x)^r.$$

(iv) $(2-3x)^{\frac{-1}{3}} = 2^{\frac{-1}{3}}\left(1-\dfrac{3x}{2}\right)^{\frac{-1}{3}} = \dfrac{1}{\sqrt[3]{2}}\left(1-\dfrac{3x}{2}\right)^{\frac{-1}{3}}$

Hence, the general term of $(2-3x)^{\frac{-1}{3}}$ is

$$T_{r+1} = \frac{1}{\sqrt[3]{2}}\cdot\frac{p(p+q)(p+2q)...(p+(r-1)q)}{r!}\left(\frac{X}{q}\right)^r$$

where $p=1,\ q=3$ and $X = \dfrac{3x}{2}$

$$\text{Thus, } T_{r+1} = \frac{1}{\sqrt[3]{2}}\cdot\frac{1\cdot4\cdot7\cdots(3r-2)}{r!}\left(\frac{x}{2}\right)^r.$$

5. Problem : *Find the coefficient of* x^{12} in $\dfrac{(1+3x)}{(1-4x)^4}$

Solution

$$\frac{(1+3x)}{(1-4x)^4} = (1+3x)(1-4x)^{-4}$$

$$= (1+3x)\left(\sum_{r=0}^{\infty} {}^{(n+r-1)}C_r\ X^r\right) \text{ where } X = 4x \text{ and } n = 4$$

$$= (1+3x)\left(\sum_{r=0}^{\infty} {}^{(4+r-1)}C_r\ (4x)^r\right) \quad = (1+3x)\left(\sum_{r=0}^{\infty} {}^{(r+3)}C_3\ .4^r\ .\ x^r\right)$$

In this expansion, the coefficient of x^{12}

$$= 1\left\{{}^{(12+3)}C_3\ .4^{12}\right\} + 3\left\{{}^{(11+3)}C_3 .4^{11}\right\}$$

$$= {}^{15}C_3 .4^{12} + 3.\left({}^{14}C_3\right).4^{11}$$

$$= 455(4^{12}) + (1092)\ 4^{11}$$

$$= 728\ (4^{12}).$$

6. Problem : *Find the coefficient of* x^6 *in the expansion of* $(1-3x)^{\frac{-2}{5}}$.

Solution

$$(1-3x)^{\frac{-2}{5}} = (1-X)^{\frac{-p}{q}} \quad \text{where } X=3x,\ p=2,\ q=5.$$

In this expansion, the general term

$$T_{r+1} = \frac{p(p+q)\ ...(p+(r-1)q)}{r!}\left(\frac{X}{q}\right)^r$$

$$= \frac{2\ (7)\ (12)...(5r-3)}{r!}\left(\frac{3x}{5}\right)^r.$$

Substituting $r = 6$, we get the coefficient of x^6 in the expansion of $(1-3x)^{\frac{-2}{5}}$

$$= \frac{2\ (7)\ (12)...(27)}{6!}\left(\frac{3}{5}\right)^6.$$

7. Problem : *Find the sum of the infinite series*

$$1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2.5}{3.6}\left(\frac{1}{2}\right)^2+\frac{2.5.8}{3.6.9}\left(\frac{1}{2}\right)^3+\cdots\infty$$

Solution : The given series can be written as

$$S=1+\frac{2}{1}\cdot\frac{1}{6}+\frac{2.5}{1.2}\left(\frac{1}{6}\right)^2+\frac{2.5.8}{1.2.3}\left(\frac{1}{6}\right)^3+\cdots\infty$$

The series on the right is of the form

$$1+\frac{p}{1!}\cdot\frac{x}{q}+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+\frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3+\cdots\infty$$

where $p=2,\ p+q=5$ and $\frac{x}{q}=\frac{1}{6}$ or $p=2,\ q=3,\ x=\frac{1}{2}$.

Hence, by the binomial theorem for rational index 6.2.1(6),

$$S=(1-x)^{\frac{-p}{q}}=\left(1-\frac{1}{2}\right)^{\frac{-2}{3}}=2^{\frac{2}{3}}=\sqrt[3]{4}.$$

8. Problem : *Find the sum of the series*

$$\frac{3.5}{5.10}+\frac{3.5.7}{5.10.15}+\frac{3.5.7.9}{5.10.15.20}+\cdots\infty.$$

Solution: Write $S=\frac{3.5}{5.10}+\frac{3.5.7}{5.10.15}+\frac{3.5.7.9}{5.10.15.20}+\cdots\infty$

On adding $1+\frac{3}{5}$ both sides, we get

$$1+\frac{3}{5}+S=1+\frac{3}{5}+\frac{3.5}{5.10}+\frac{3.5.7}{5.10.15}+\cdots\infty$$

$$=1+\frac{3}{1!}\left(\frac{1}{5}\right)+\frac{3.5}{2!}\left(\frac{1}{5}\right)^2+\frac{3.5.7}{3!}\left(\frac{1}{5}\right)^3+\cdots\infty$$

$$=1+\frac{p}{1!}\cdot\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+\frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3+\cdots$$

$$\left(\text{where } p=3,\ \ p+q=5 \text{ and } \frac{x}{q}=\frac{1}{5}\right)$$

$$= (1-x)^{\frac{-p}{q}}, \text{ where } p=3,\ q=2 \text{ and } x=\frac{2}{5}$$

$$= \left(1-\frac{2}{5}\right)^{\frac{-3}{2}} = \left(\frac{3}{5}\right)^{\frac{-3}{2}} = \left(\frac{5}{3}\right)^{\frac{3}{2}}$$

$$\Rightarrow \text{ S} = \frac{5\sqrt{5}}{3\sqrt{3}} - \frac{8}{5}.$$

9. Problem : *If* $x = \frac{1}{5} + \frac{1.3}{5.10} + \frac{1.3.5}{5.10.15} + \cdots\infty$, *then find* $3x^2 + 6x$

Solution : $x = \frac{1}{5} + \frac{1.3}{5.10} + \frac{1.3.5}{5.10.15} + \cdots\infty$

$$\Rightarrow 1+x = 1+\frac{1}{5} + \frac{1.3}{5.10} + \frac{1.3.5}{5.10.15} + \cdots\infty$$

$$= 1+\frac{1}{1!}\left(\frac{1}{5}\right) + \frac{1.3}{2!}\left(\frac{1}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{1}{5}\right)^3 + \cdots\infty$$

$$= 1+\frac{p}{1!}.\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 + \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \cdots\infty$$

where $p=1,\ p+q = 3$ and $\frac{x}{q} = \frac{1}{5}$

$$= (1-x)^{\frac{-p}{q}}, \text{ where } p=1,\ q=2 \text{ and } x=\frac{2}{5}$$

$$= \left(1-\frac{2}{5}\right)^{\frac{-1}{2}} = \left(\frac{3}{5}\right)^{\frac{-1}{2}} = \sqrt{\frac{5}{3}}$$

$$\Rightarrow (1+x)^2 = \frac{5}{3} \Rightarrow x^2+2x+1 = \frac{5}{3} \Rightarrow 3x^2+6x+3=5$$

Therefore, $3x^2+6x = 2$.

Exercise 6(b)

I. 1. Find the set of values of x for which the binomial expansions of the following are valid.

(i) $(2+3x)^{\frac{-2}{3}}$ (ii) $(5+x)^{\frac{3}{2}}$ (iii) $(7+3x)^{-5}$ (iv) $\left(4-\frac{x}{3}\right)^{\frac{-1}{2}}$

2. Find the

(i) 6^{th} term of $\left(1+\frac{x}{2}\right)^{-5}$ (ii) 7^{th} term of $\left(1-\frac{x^2}{3}\right)^{-4}$

(iii) 10^{th} term of $(3-4x)^{\frac{-2}{3}}$ (iv) 5^{th} term of $\left(7+\frac{8y}{3}\right)^{\frac{7}{4}}$

3. Write down the first 3 terms in the expansion of

(i) $(3+5x)^{\frac{-7}{3}}$ (ii) $(1+4x)^{-4}$ (iii) $(8-5x)^{\frac{2}{3}}$ (iv) $(2-7x)^{\frac{-3}{4}}$

4. Find the general term ($(r+1)^{th}$ term) in the expansion of

(i) $(4+5x)^{\frac{-3}{2}}$ (ii) $\left(1-\frac{5x}{3}\right)^{-3}$ (iii) $\left(1+\frac{4x}{5}\right)^{\frac{5}{2}}$ (iv) $\left(3-\frac{5x}{4}\right)^{\frac{-1}{2}}$

II. 1. Find the coefficient of x^{10} in the expansion of $\frac{1+2x}{(1-2x)^2}$.

2. Find the coefficient of x^4 in the expansion of $(1-4x)^{\frac{-3}{5}}$.

3. (i) Find the coefficient of x^5 in $\frac{(1-3x)^2}{(3-x)^{\frac{3}{2}}}$.

(ii) Find the coefficient of x^8 in $\frac{(1+x)^2}{(1-\frac{2}{3}x)^3}$.

(iii) Find the coefficient of x^7 in $\frac{(2+3x)^3}{(1-3x)^4}$.

4. Find the coefficient of x^3 in the expansion of $\frac{(1+3x^2)^{\frac{3}{2}}}{(3+4x)^{\frac{1}{3}}}$.

III. 1. Find the sum of the infinite series

(i) $1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\cdots$ (ii) $1-\frac{4}{5}+\frac{4.7}{5.10}-\frac{4.7.10}{5.10.15}+\cdots$

(iii) $\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\cdots$ (iv) $\frac{3}{4.8}-\frac{3.5}{4.8.12}+\frac{3.5.7}{4.8.12.16}-\cdots$

2. If $t=\frac{4}{5}+\frac{4.6}{5.10}+\frac{4.6.8}{5.10.15}+\ldots\infty$, then prove that $9t=16$.

3. If $x=\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\frac{1.3.5.7}{3.6.9.12}+\ldots$, then prove that $9x^2+24x=11$.

4. If $x=\frac{5}{(2!).3}+\frac{5.7}{(3!).3^2}+\frac{5.7.9}{(4!).3^3}+\cdots$, then find the value of x^2+4x.

5. Find the sum to infinite terms of the series

$$\frac{7}{5}\left(1+\frac{1}{10^2}+\frac{1.3}{1.2}\cdot\frac{1}{10^4}+\frac{1.3.5}{1.2.3}\cdot\frac{1}{10^6}+\ldots\right)$$

6. Show that for any non zero rational number x.

$$1+\frac{x}{2}+\frac{x(x-1)}{2.4}+\frac{x(x-1)(x-2)}{2.4.6}+\cdots$$

$$=1+\frac{x}{3}+\frac{x(x+1)}{3.6}+\frac{x(x+1)(x+2)}{3.6.9}+\cdots$$

6.3 Approximations using Binomial Theorem

If n is a rational number and x is any real number with $|x|<1$, we have learnt, in the previous section 6.2, that

$$(1+x)^n=1+\frac{n}{1}x+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\cdots$$

The series on the right hand side of the above equation contains infinitely many terms unless n is a non negative integer. We can use this binomial theorem to find approximate values of irrational numbers. If $|x|<1$, as r increases, the value of $|x|^r$ becomes smaller and smaller and hence we can neglect the terms in R.H.S. on or after certain stage to get an approximate value of $(1+x)^n$. The accuracy of this approximate value of $(1+x)^n$ depends on the number of terms (in R.H.S.) taken into consideration. For finding an approximate value of $(1+x)^n$, we make use of the results of the previous section.

The following observations will help in finding an approximate value of $(1+x)^n$ (when $|x|<1$)

(i) If x^2 and higher powers of x can be neglected then

$$\boxed{(1+x)^n \simeq 1+nx}$$

(here the symbol $\simeq$ stands for the phrase 'is approximately equal to')

(ii) If x^3 and higher powers of x can be neglected then

$$\boxed{(1+x)^n \simeq 1+nx+\frac{n(n-1)}{2!}x^2}$$

(iii) If x^4 and higher powers of x can be neglected then

$$\boxed{(1+x)^n \simeq 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3}$$

We illustrate this procedure through the following examples.

6.3.1 Example

Find an approximate value of (i) $\dfrac{1}{\sqrt[3]{999}}$ (ii) $(627)^{\frac{1}{4}}$

corrected to 5 decimal places

Solution

(i) $$\frac{1}{\sqrt[3]{999}} = (999)^{\frac{-1}{3}} = (1000-1)^{\frac{-1}{3}}$$

$$= (1000)^{\frac{-1}{3}}\left(1-\frac{1}{1000}\right)^{\frac{-1}{3}}$$

$$= \frac{1}{10}\left\{1+\frac{1}{3}.\frac{1}{1000}+\frac{\frac{1}{3}.\frac{4}{3}}{2!}\left(\frac{1}{1000}\right)^2+\cdots\right\}$$

$$= \frac{1}{10}\{1+0.000333+0.00000022+\ldots\} \simeq \frac{1}{10}\{1.00033322\}$$

$\simeq$ 0.10003 (corrected to 5 decimal places). (approximately)

(ii) $$(627)^{\frac{1}{4}} = (625+2)^{\frac{1}{4}} = (625)^{\frac{1}{4}}\left(1+\frac{2}{625}\right)^{\frac{1}{4}}$$

$$= 5\left\{1+\frac{1}{4}\cdot\frac{2}{625}+\frac{\frac{1}{4}\cdot\left(\frac{-3}{4}\right)}{2!}\left(\frac{2}{625}\right)^2+\ldots\right\}$$

$$= 5\left\{1+0.0008-\frac{96}{10^8}+\cdots\right\}$$

$$\simeq 5(1.0008) = 5.004 \text{ (corrected to 5 decimal places).}$$

6.3.2 Example: If $|x|$ is so small that x^3 and higher powers of x can be neglected, find approximate value of

$$\frac{(4-7x)^{\frac{1}{2}}}{(3+5x)^3}.$$

Solution

$$\frac{(4-7x)^{\frac{1}{2}}}{(3+5x)^3} = \frac{4^{\frac{1}{2}}\left(1-\frac{7x}{4}\right)^{\frac{1}{2}}}{3^3\left(1+\frac{5x}{3}\right)^3} = \frac{2}{27}\left(1-\frac{7x}{4}\right)^{\frac{1}{2}}\left(1+\frac{5x}{3}\right)^{-3}$$

$$\simeq \frac{2}{27}\left(1+\frac{\frac{1}{2}}{1!}\left(\frac{-7x}{4}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{-7x}{4}\right)^2\right)\left(1+\frac{-3}{1!}\left(\frac{5x}{3}\right)+\frac{(-3)(-4)}{2!}\left(\frac{5x}{3}\right)^2\right)$$

$$= \frac{2}{27}\left(1-\frac{7x}{8}-\frac{49}{128}x^2\right)\left(1-5x+\frac{50x^2}{3}\right)$$ (after neglecting x^3 and higher powers of x)

$$= \frac{2}{27}\left(1-5x+\frac{50x^2}{3}-\frac{7x}{8}+\frac{35}{8}x^2-\frac{49}{128}x^2\right)$$ (again by neglecting x^3 and x^4 terms)

$$= \frac{2}{27}\left(1-\frac{47}{8}x+\frac{7933}{384}x^2\right).$$

6.3.3 Solved Problems

1. Problem : Find an approximate value of $\sqrt[6]{63}$ correct to 4 decimal places.

Solution

$$\sqrt[6]{63} = (63)^{\frac{1}{6}} = (64-1)^{\frac{1}{6}} = (64)^{\frac{1}{6}} \left(1-\frac{1}{64}\right)^{\frac{1}{6}}$$

$$= 2\left(1-(0.5)^6\right)^{\frac{1}{6}} \quad = 2\left((1-\frac{\frac{1}{6}}{1!}(0.5)^6+\frac{\frac{1}{6}\left(\frac{1}{6}-1\right)}{2!}(0.5)^{12}-\cdots\right)$$

$$\simeq 2(1-0.0026041) \quad = 2(0.9973959) \quad = 1.9947918$$

$$\simeq 1.9948 \quad \text{(corrected to 4 decimals).}$$

2. Problem : If $|x|$ is so small that x^2 and higher powers of x may be neglected, then find an approximate value of

$$\frac{\left(1+\frac{3x}{2}\right)^{-4}(8+9x)^{\frac{1}{3}}}{(1+2x)^2}.$$

Solution

$$\frac{\left(1+\frac{3x}{2}\right)^{-4}(8+9x)^{\frac{1}{3}}}{(1+2x)^2} = \left(1+\frac{3x}{2}\right)^{-4} . 8^{\frac{1}{3}}\left(1+\frac{9x}{8}\right)^{\frac{1}{3}}(1+2x)^{-2}$$

$$\simeq 2\left(1-\frac{4}{1}\left(\frac{3x}{2}\right)\right)\left(1+\frac{1}{3}.\frac{9x}{8}\right)(1-2(2x)) \quad \text{(after neglecting } x^2 \text{ and higher powers of } x\text{)}$$

$$\simeq 2(1-6x)\left(1+\frac{3x}{8}\right)(1-4x)$$

$$\simeq 2\left(1-\frac{45}{8}x\right)(1-4x) \quad \text{(neglecting } x^2 \text{ term)}$$

$$\simeq 2\left(1-\frac{77}{8}x\right) \quad \text{(again neglecting } x^2 \text{ term)}$$

3. Problem : If $|x|$ is so small that x^4 and higher powers of x may be neglected, then find an approximate value of

$$\sqrt[4]{x^2+81} - \sqrt[4]{x^2+16}$$

Solution

$$\sqrt[4]{x^2+81} - \sqrt[4]{x^2+16} = \left(81+x^2\right)^{\frac{1}{4}} - \left(16+x^2\right)^{\frac{1}{4}}$$

$$= 81^{\frac{1}{4}}\left(1+\frac{x^2}{81}\right)^{\frac{1}{4}} - 16^{\frac{1}{4}}\left(1+\frac{x^2}{16}\right)^{\frac{1}{4}}$$

$$\simeq 3\left(1+\frac{1}{4}.\frac{x^2}{81}\right) - 2.\left(1+\frac{1}{4}.\frac{x^2}{16}\right)$$

$$\simeq 1-\frac{19}{864}x^2 \quad \text{(after neglecting } x^4 \text{ and higher powers of } x\text{)}$$

4. Problem : Suppose that x and y are positive and x is very small when compared to y. Then find an approximate value of $\left(\frac{y}{y+x}\right)^{\frac{3}{4}} - \left(\frac{y}{y+x}\right)^{\frac{4}{5}}$.

Solution

$$\left(\frac{y}{y+x}\right)^{\frac{3}{4}} - \left(\frac{y}{y+x}\right)^{\frac{4}{5}} = \left(\frac{1}{1+\frac{x}{y}}\right)^{\frac{3}{4}} - \left(\frac{1}{1+\frac{x}{y}}\right)^{\frac{4}{5}} = \left(1+\frac{x}{y}\right)^{\frac{-3}{4}} - \left(1+\frac{x}{y}\right)^{\frac{-4}{5}}$$

$$\simeq \left(1-\frac{3}{4}.\frac{x}{y}+\frac{\frac{3}{4}\left(\frac{3}{4}-1\right)}{2!}\left(\frac{x}{y}\right)^2\right) - \left(1-\frac{4}{5}\frac{x}{y}+\frac{\frac{4}{5}\left(\frac{4}{5}-1\right)}{2!}\left(\frac{x}{y}\right)^2\right)$$

(neglecting $\left(\frac{x}{y}\right)^3$ and higher powers of $\frac{x}{y}$)

$$= \left(1-\frac{3}{4}.\frac{x}{y}-\frac{3}{32}\left(\frac{x}{y}\right)^2\right) - \left(1-\frac{4}{5}\frac{x}{y}-\frac{2}{25}\left(\frac{x}{y}\right)^2\right)$$

$$\simeq \frac{1}{20}\left(\frac{x}{y}\right) - \frac{11}{800}\left(\frac{x}{y}\right)^2.$$

5. Problem : Expand $5\sqrt{5}$ in increasing powers of $\frac{4}{5}$.

Solution

$$5\sqrt{5} = 5^{\frac{3}{2}} = \left(\frac{1}{5}\right)^{\frac{-3}{2}} = \left(1-\frac{4}{5}\right)^{\frac{-3}{2}}$$

$$= 1+\frac{\frac{3}{2}}{1!}\frac{4}{5}+\frac{\frac{3}{2}\cdot\frac{5}{2}}{2!}\left(\frac{4}{5}\right)^2+\ldots+\frac{\frac{3}{2}\cdot\frac{5}{2}\ldots(\frac{3}{2}+r-1)}{r!}\left(\frac{4}{5}\right)^r+\cdots\infty$$

$$= 1+\frac{3}{1!.2}\frac{4}{5}+\frac{3.5}{2!.2^2}\left(\frac{4}{5}\right)^2+\cdots+\frac{3.5\ldots(2r+1)}{r!2^r}\left(\frac{4}{5}\right)^r+\cdots\infty$$

Exercise 6(c)

I. 1. Find an approximate value of the following corrected to 4 decimal places.

(i) $\sqrt[5]{242}$ (ii) $\sqrt[7]{127}$ (iii) $\sqrt[5]{32.16}$

(iv) $\sqrt{199}$ (v) $\sqrt[3]{1002}-\sqrt[3]{998}$ (vi) $(1.02)^{3/2}-(0.98)^{3/2}$

2. If $|x|$ is so small that x^2 and higher powers of x may be neglected, then find approximate values of the following.

(i) $\dfrac{(4+3x)^{\frac{1}{2}}}{(3-2x)^2}$ (ii) $\dfrac{\left(1-\frac{2x}{3}\right)^{\frac{3}{2}}(32+5x)^{\frac{1}{5}}}{(3-x)^3}$ (iii) $\sqrt{4-x}\left(3-\frac{x}{2}\right)^{-1}$

(iv) $\dfrac{\sqrt{4+x}+\sqrt[3]{8+x}}{(1+2x)+(1-2x)^{\frac{-1}{3}}}$ (v) $\dfrac{(8+3x)^{\frac{2}{3}}}{(2+3x)\sqrt{4-5x}}$

3. Suppose s and t are positive and t is very small when compared to s. Then find an approximate value of

$$\left(\frac{s}{s+t}\right)^{\frac{1}{3}}-\left(\frac{s}{s-t}\right)^{\frac{1}{3}}.$$

4. Suppose p, q are positive and p is very small when compared to q. Then find an approximate value of

$$\left(\frac{q}{q+p}\right)^{\frac{1}{2}}+\left(\frac{q}{q-p}\right)^{\frac{1}{2}}.$$

5. By neglecting x^4 and higher powers of x, find an approximate value of

$$\sqrt[3]{x^2+64}-\sqrt[3]{x^2+27}.$$

6. Expand $3\sqrt{3}$ in increasing powers of $\frac{2}{3}$.

Key Concepts

If a, x are real numbers and n is a positive integer, then

- $(x+a)^n = {}^nC_0 . x^n . a^o + {}^nC_1 . x^{n-1} . a^1 + {}^nC_2 . x^{n-2} . a^2 + \cdots + {}^nC_r . x^{n-r} . a^r + \cdots + {}^nC_n . x^0 . a^n$
- $(x-a)^n = {}^nC_0 . x^n . a^o - {}^nC_1 . x^{n-1} . a^1 + {}^nC_2 . x^{n-2} . a^2 - \cdots + (-1)^r\, {}^nC_r . x^{n-r} . a^r + \cdots + (-1)^n\, {}^nC_n . x^0 . a^n$
- Each of the above two expansions contain $(n+1)$ terms in R.H.S.
- The general term of $(x+a)^n$ is $T_{r+1} = {}^nC_r . x^{n-r} . a^r \ (0 \le r \le n)$
- If n is fixed we write ${}^nC_r = C_r$ and $C_0, C_1, C_2, \ldots, C_n$ are called the binomial coefficients.
- (i) $C_0 + C_1 + C_2 + \cdots + C_n = 2^n$

  (ii) $C_0 - C_1 + C_2 - C_3 + \cdots + (-1)^n\, C_n = 0$

  (iii) $C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots = 2^{n-1}$
- (i) $\sum_{r=0}^{n} C_r = 2^n$ (ii) $\sum_{r=1}^{n} r . C_r = n . 2^{n-1}$

  (iii) $\sum_{r=2}^{n} r(r-1) . C_r = n(n-1) . 2^{n-2}$ (iv) $\sum_{r=1}^{n} r^2 . C_r = n(n+1) . 2^{n-2}$
- (i) If n is even then the expansion of $(x+a)^n$ has only one middle term. It is

  $$T_{\frac{n}{2}+1} = {}^nC_{\frac{n}{2}} . x^{\frac{n}{2}} . a^{\frac{n}{2}}$$

  (ii) If n is odd, then the expansion of $(x+a)^n$ has two middle terms. They are

  $$T_{\frac{n+1}{2}} = {}^nC_{\left(\frac{n-1}{2}\right)} . x^{\frac{n+1}{2}} . a^{\frac{n-1}{2}} \text{ and } T_{\frac{n+3}{2}} = {}^nC_{\left(\frac{n+1}{2}\right)} . x^{\frac{n-1}{2}} . a^{\frac{n+1}{2}}$$
- Numerically greatest term in the expansion of $(1+x)^n$:

  (i) If $\frac{(n+1)|x|}{1+|x|}$ is not an integer and if its integral part $\left[\frac{(n+1)|x|}{1+|x|}\right] = r$, a positive integer then t_{r+1} is the numerically greatest term in the expansion of $(1+x)^n$.

  (ii) If $\frac{(n+1)|x|}{1+|x|}$ is positive integer, say m, then $|T_m| = |T_{m+1}|$ and hence T_m and T_{m+1} both are numerically greatest terms in the expansion of $(1+x)^n$.

❖ If x is a real number such that $|x|<1$ and p, q are positive integers then

(i) $(1+x)^{\frac{p}{q}} = 1 + \frac{p}{1!}.\frac{x}{q} + \frac{p(p-q)}{2!}\left(\frac{x}{q}\right)^2 + \dots + \frac{p(p-q)\dots(p-(r-1)q)}{r!}\left(\frac{x}{q}\right)^r + \dots \infty$

(ii) $(1-x)^{\frac{p}{q}} = 1 - \frac{p}{1!}\frac{x}{q} + \frac{p(p-q)}{2!}\left(\frac{x}{q}\right)^2 - \dots + (-1)^r \frac{p(p-q)\dots(p-(r-1)q)}{r!}\left(\frac{x}{q}\right)^r + \dots \infty$

(iii) $(1+x)^{\frac{-p}{q}} = 1 - \frac{p}{1!}\frac{x}{q} + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 - \dots + (-1)^r \frac{p(p+q)\dots(p+(r-1)q)}{r!}\left(\frac{x}{q}\right)^r + \dots \infty$

(iv) $(1-x)^{\frac{-p}{q}} = 1 + \frac{p}{1!}\frac{x}{q} + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 + \dots + \frac{p(p+q)\dots(p+(r-1)q)}{r!}\left(\frac{x}{q}\right)^r + \dots \infty$

❖ If n is a positive integer and x is a real number such that $|x|<1$, then

(i) $(1+x)^{-n} = 1 - \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 - \dots + (-1)^r . \frac{n(n+1)\cdots(n+r-1)}{r!}x^r + \dots \infty$

$$= \sum_{r=0}^{\infty} (-1)^r . {}^{(n+r-1)}C_r . x^r$$

(ii) $(1-x)^{-n} = 1 + \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 + \dots + \frac{n(n+1)\dots(n+r-1)}{r!}x^r + \dots \infty$

$$= \sum_{r=0}^{\infty} . {}^{(n+r-1)}C_r . x^r$$

❖ (i) If $|x|$ is so small that x^2 and higher powers of x may be neglected then

$$(1+x)^n \simeq 1 + nx \cdot$$

(ii) If $|x|$ is so small that x^3 and higher powers of x may be neglected then

$$(1+x)^n \simeq 1 + nx + \frac{n(n-1)}{2!}x^2 .$$

(iii) If $|x|$ is so small that x^4 and higher powers of x may be neglected then

$$(1+x)^n \simeq 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 .$$

Historical Note

In medieval India, various rules for obtaining binomial coefficients have been described by *Varahanka* (between 600-800) *Jayadeva* (around 9th Century A.D.) and others.

The triangular arrangement of the coefficients in the expansion of $(x+y)^n$ for $0 \le n \le 7$, were also found in the work of Chinese mathematician *Chi.Shi.Kie* in 1303. The term binomial coefficients was first used by the German mathematician, *Michael Stifel* (1486-1567) in his work in the year 1544. The coefficients of the expansion of $(a+b)^n$ were also given by *Bombelli* (1572) for n = 1, 2,....7, and *Oligarch* (1631) for $n=1,2,.......,10.$. Though the arithmetic triangle (given at the beginning of this chapter) was mentioned by the Indian '*Pingala*' in his book "*Chandassastra*" under the name '*Meru-Prastara*' as early as 200 B.C. it was popularized by the French mathematician *Blaise Pascal* (1623-1662) and hence it was named as '*Pascal's triangle*'. The present form of the binomial theorem for the positive integral values of n appeared first in the book *'Traite du triangle arithmetique'*, written by *Pascal*.

Answers

Exercise 6(a)

I. 1. (i) $\sum_{r=0}^{7} {}^7C_r.(4x)^{7-r}.(5y)^r$ (ii) $\sum_{r=0}^{5} {}^5C_r.\left(\frac{2}{3}x\right)^{5-r}\left(\frac{7}{4}y\right)^r$

(iii) $\sum_{r=0}^{6} (-1)^r\ {}^6C_r.\left(\frac{2p}{5}\right)^{6-r}\left(\frac{3q}{7}\right)^r$

(iv) $81+108x-54x^2-96x^3+19x^4+32x^5-6x^6-4x^7+x^8$

2. (i) $189\,x^4\,y^5$ (ii) $280.(12)^5.\,x^4\,y^6$

(iii) $\frac{-(2002).3^5.5^9}{4^5}.p^5.q^9$ (iv) ${}^8C_{r-1}.\left(\frac{3a}{5}\right)^{9-r}.\left(\frac{5b}{7}\right)^{r-1}$

3. (i) 10 (ii) 15 (iii) 36

4. 25 5. 2^{38} 6. 2

II. 1. (i) 405×4^8 (ii) $286\times2^{10}\times3^3$

(iii) $-126\times7^4\times2^5$ (iv) $\frac{-4375}{324}$

2. (i) $T_3 = \frac{80}{729}$ (ii) $T_{11} = {}^{25}C_{10} \,.\, 3^{15}.5^{10}$

(iii) 0 (iv) $T_7 = \frac{3^7 \times 5^5 \times 7}{2^7}$

3. (i) $T_6 = -{}^{10}C_5 \,.\left(\frac{6}{7}\right)^5 .\, x^5 .\, y^5$

(ii) $T_6 = 77 \times 2^8 \times 3^6 \times a^6 b^5$ and $T_7 = 77 \times 2^5 \times 3^7 \times a^5 b^6$

(iii) $T_9 = {}^{17}C_8 \,.\, 4^9.5^8.x^{42}$ and $T_{10} = {}^{17}C_9 \,.\, 4^8.5^9.x^{43}$

(iv) $T_{11} = {}^{20}C_{10}\ 15^{10}\ a^{10}$

4. (i) $T_{12} = {}^{15}C_{11} \,.\, \frac{(21)^{11}}{2^3}$ (ii) $T_{11} = {}^{12}C_{10} \,.\left(\frac{3}{2}\right)^2 .\left(\frac{20}{3}\right)^{10}.$

(iii) $T_{10} = (-1)\ {}^{13}C_9\ (12)^4\ (30)^9\ 8,\ T_{11} = 143 \times 2^{17} \times 3^{13} \times 5^{10}$

(iv) $T_{11} = {}^{15}C_{11}\left(\frac{28}{5}\right)^{11} 3^4$

6. (i) 120 (ii) ${}^{2n}C_{n+3}$ (iii) $(n^2 + 9n + 16)2^{n-2}$.

(iv) 3^{n+1} (v) 3^n

10. (i) 2^{13} (ii) 2^{13} 11. 20 12. -2 13. 0, 3

III. 2. 9 3. $a = 2,\ x = 3,\ n = 5$ 5. $91 \times 36 \times (2^{10} + 3^{10})$ 10. ${}^{2n}C_n$

13. $\frac{18\sqrt[3]{2} - 22}{3}$ 17. 97

Exercise 6(b)

I. 1. (i). $\left(\frac{-2}{3}, \frac{2}{3}\right)$ (ii) $(-5, 5)$ (iii) $\left(\frac{-7}{3}, \frac{7}{3}\right)$ (iv) $(-12, 12)$

2. (i). $\frac{-63}{16}x^5$ (ii) $\frac{28}{243}x^{12}$ (iii) $\frac{2.5.8...}{\sqrt[3]{9}.9!}\left(\frac{4x}{9}\right)^9$ (iv) $7^{\frac{7}{4}}(70).\left(\frac{y}{21}\right)^4$

3. (i). $3^{\frac{-7}{3}}, \frac{-3^{\frac{-7}{3}}.35x}{9}, \frac{3^{\frac{-7}{3}}.875}{81}x^2$ (ii) $1, -16x, 160x^2$

(iii). $4, \frac{-5}{3}x, \frac{-25}{144}x^2$ (iv) $1, \frac{21}{8}x, \frac{1029}{128}x^2$

4. (i). $(-1)^r.\dfrac{3.7.5..(2r+1)}{r!}.\dfrac{(5x)^r}{8^{r+1}}$ (ii) ${}^{r+2}C_2\left(\dfrac{5x}{3}\right)^r$

(iii). $(-1)^{r-1}.\dfrac{5.3.1.1.3.5..(2r-7)}{r!}\left(\dfrac{2x}{5}\right)^r$ (iv) $\dfrac{1}{\sqrt{3}}\left(\dfrac{1.3.5...(2r-1)}{r!}\right).\left(\dfrac{5x}{24}\right)^r$

II. 1. 21×2^{10} 2. $\dfrac{234\times 256}{625}$

3. (i) $\dfrac{3857}{\sqrt{3}(12)^4}$ (ii) $\dfrac{2048}{243}$

(iii) $8\left({}^{10}C_3.3^7\right)+36\left({}^{9}C_3.3^6\right)+54\left({}^{8}C_3.3^5\right)+27\left({}^{7}C_3.3^4\right)$

4. $\dfrac{-5270}{2187\sqrt[3]{3}}$

III. 1. (i) $\sqrt{3}$ (ii) $\dfrac{5^{4/3}}{16}$

(iii) $2\sqrt{2}-1$ (iv) $\sqrt{\dfrac{2}{3}-\dfrac{3}{4}}$

4. 23 5. $\sqrt{2}$

Exercise 6(c)

I. 1. (i) 2.9975 (ii) 1.9978 (iii) 2.002 (iv) 14.1068 (v) 0.0133 (vi) 0.059999

2. (i) $\dfrac{24+41x}{108}$ (ii) $\dfrac{2}{27}\left(1+\dfrac{x}{32}\right)$ (iii) $\dfrac{2}{3}\left(1+\dfrac{x}{24}\right)$ (iv) $\dfrac{4-5x}{2}$ (v) $1-\dfrac{5x}{8}$

3. $-\dfrac{2}{3}\dfrac{t}{s}-\dfrac{28}{81}\left(\dfrac{t}{s}\right)^3$ 4. $2+\dfrac{3}{4}\left(\dfrac{p}{q}\right)^2$ 5. $1-\dfrac{7}{432}x^2$

6. $1+3\left(\dfrac{1}{3}\right)+\dfrac{3.5}{2!}\left(\dfrac{1}{3}\right)^2+\dfrac{3.5.7}{3!}\left(\dfrac{1}{3}\right)^3+\cdots+\dfrac{3.5.7...(2r+1)}{(r+1)!}\left(\dfrac{1}{3}\right)^r+\cdots$

Chapter 7

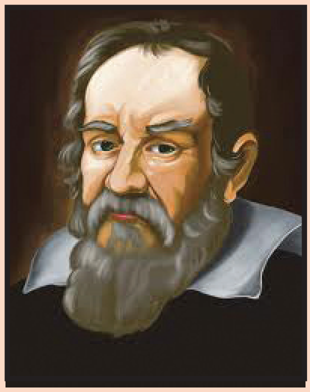
Partial Fractions

" Measure what is measurable and make measurable what is not so"

- Galilei Galileo

Introduction

In chapter four we have defined a polynomial in x of degree n and learnt the methods of finding roots of polynomial equations. We frequently come across quotient of polynomials, which we may call polynomial fractions or rational fractions or simply fractions. We often require expansion of these fractions in power series. Therefore, we have to express the fractions like $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x) \neq 0$ are polynomials as the sum of certain terms known as partial fractions. In this chapter we learn the partial fraction decomposition of a fraction. It is useful in many situations like finding particular integrals of differential equations, evaluation of integrals, expanding infinite series and in some cases, summing the infinite series etc.



Galilei Galileo

(1564 - 1642)

Galileo *of Italy was an outstanding astronomer who contributed notably to mathematics. His work on pendulum even when he was a student, got him fame. He was a professor of mathematics at Pisa and also at Padua. He is also famous for his discovery of telescope. He displayed the modern spirit of science as a harmony between experiment and theory.*

7.0 Rational Fractions

In the next section, we describe some concepts that are required for our discussion and we define the terms rational fraction, reducible and irreducible polynomial. Throughout this chapter we consider polynomials with real coefficients. Henceforth a polynomial means a polynomial with real coefficients.

7.0.1 Definition (Rational Fraction)

If $f(x)$ and $g(x)$ are two polynomials and $g(x)$ is a non-zero polynomial, then $\frac{f(x)}{g(x)}$ is called a rational fraction or polynomial fraction or simply a fraction.

Examples : $\frac{5x+1}{x^2+x-2}$ and $\frac{x^3+2x+5}{x^2-2}$ are rational fractions.

7.0.2 Definition (Proper and Improper Fractions)

A rational fraction $\frac{f(x)}{g(x)}$ is called a proper fraction if the degree of $f(x)$ is less than the degree of $g(x)$. Otherwise it is called an improper fraction.

Examples

(i) $\frac{5x+1}{x^2+x-2}$ is a proper fraction

(ii) $\frac{x^4}{x^3-3x+2}$ is an improper fraction

(iii) $\frac{x^3}{(2x-1)(x+2)(x-3)}$ is an improper fraction

7.0.3 Definition (Irreducible Polynomial)

A polynomial $f(x)$ is said to be irreducible if it cannot be expressed as a product of two polynomials $g(x)$ and $h(x)$ such that the degree of each polynomial is less than the degree of $f(x)$. If $f(x)$ is not irreducible then we say that $f(x)$ is reducible.

Examples

(i) $2x + 1$ is an irreducible polynomial.

(ii) $x^2 + x + 1$ is an irreducible polynomial

(iii) $x^3 - 6x^2 + 11x - 6$ is reducible, since

$x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$

(iv) $x^2 - 4x + 13$ is irreducible as it cannot be expressed as a product of linear polynomials whose coefficients are real. Though $x^2 - 4x + 13 = [x - (2 + 3i)]\,[x - (2 - 3i)]$, it is irreducible as the coefficients of the factors on the right hand side are not real.

7.0.4 Note

(i) Every linear polynomial is irreducible. For example, $2x + 3$ is irreducible.

(ii) If $a \neq 0$, $ax^2 + bx + c$ is irreducible, iff $b^2 - 4ac < 0$. For example, $x^2 - x + 1$ is irreducible since $b^2 - 4ac = -3 < 0$.

7.0.5 Division Algorithm for polynomials

We state the division algorithm without proof.

If $f(x)$ and $g(x)$ are two polynomials with $g(x) \neq 0$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = q(x)g(x) + r(x)$$

where either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $g(x)$.

7.0.6 Definition (Partial Fraction)

If a proper fraction is expressed as the sum of two or more proper fractions, wherein the denominators are powers of irreducible polynomials, then each proper fraction in the sum is called a partial fraction of the given fraction.

7.1 Partial Fractions of $\frac{f(x)}{g(x)}$, when $g(x)$ contains non-repeated linear factors

In earlier classes we learnt how to add two proper fractions to get another proper fraction. For example,

$\frac{3}{2x+5} + \frac{5}{x+6} = \frac{13x+43}{2x^2+17x+30}$. Here $\frac{3}{2x+5}$, $\frac{5}{x+6}$ are called partial fractions of $\frac{13x+43}{2x^2+17x+30}$.

Now we learn using some rules to express a proper fraction as a sum of two or more proper fractions. This process is known as 'resolving into partial fractions'.

Resolution of a proper fraction $\frac{f(x)}{g(x)}$ into a sum of partial fractions depends upon factorization of $g(x)$ into linear and/or irreducible (quadratic) factors. We assume that such a decomposition is possible and is unique. We list out here under some useful rules for resolution of $\frac{f(x)}{g(x)}$ into a sum of partial fractions without presenting proofs.

To find the partial fractions of $\frac{f(x)}{g(x)}$, when $g(x)$ contains non-repeated linear factors, we use the following rule.

7.1.1 Rule I

Let $\frac{f(x)}{g(x)}$ be a proper fraction. To each non repeated factor $(ax+b)$ of $g(x)$ there will be a partial fraction of the form $\frac{A}{ax+b}$ where A is a non zero real number, to be determined.

7.1.2 Solved Problems

1. Problem : *Resolve* $\frac{5x+1}{(x+2)(x-1)}$ *into partial fractions.*

Solution : Let $\frac{5x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$,

where A and B are non-zero real numbers to be determined.

Then $\frac{5x+1}{(x+2)(x-1)} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$

$\therefore$ $A(x-1)+B(x+2) = 5x+1$. ... (1)

Putting $x=1$ in (1), we get $3B = 5+1$ i.e., $B = 2$

Putting $x=-2$ in (1), we get $-3A = -9$ i.e., $A = 3$

$$\therefore \frac{5x+1}{(x+2)(x-1)} = \frac{3}{x+2} + \frac{2}{x-1}.$$

2. Problem : *Resolve* $\dfrac{2x+3}{5(x+2)(2x+1)}$ *into partial fractions.*

Solution : Let $\dfrac{2x+3}{(x+2)(2x+1)} = \dfrac{A}{x+2} + \dfrac{B}{2x+1}$,

where A and B are non-zero real numbers, to be determined.

$$\therefore \frac{2x+3}{(x+2)(2x+1)} = \frac{A(2x+1)+B(x+2)}{(x+2)(2x+1)}$$

$$\therefore\ 2x+3 = A(2x+1)+B(x+2) \qquad \text{... (1)}$$

Putting $x = -2$ in (1), we get $-1 = -3A$ so that $A = \dfrac{1}{3}$

Putting $x = -\dfrac{1}{2}$ in (1), we get $B = \dfrac{4}{3}$

$$\therefore \frac{2x+3}{(x+2)(2x+1)} = \frac{1}{3(x+2)} + \frac{4}{3(2x+1)}$$

$$\therefore \frac{2x+3}{5(x+2)(2x+1)} = \frac{1}{5}\left\{\frac{1}{3(x+2)} + \frac{4}{3(2x+1)}\right\}$$

$$= \frac{1}{15(x+2)} + \frac{4}{15(2x+1)}.$$

3. Problem : *Resolve* $\dfrac{13x+43}{2x^2+17x+30}$ *into partial fractions.*

Solution : We have $2x^2 + 17x + 30 = (2x+5)(x+6)$

$$\therefore \frac{13x+43}{2x^2+17x+30} = \frac{13x+43}{(2x+5)(x+6)}.$$

Now, let $\dfrac{13x+43}{(2x+5)(x+6)} = \dfrac{A}{2x+5} + \dfrac{B}{x+6}$

$$\therefore\ 13x + 43 = A(x+6) + B(2x+5)$$

$$= (A+2B)x + (6A+5B)$$

Comparing the coefficients of like powers of x, we have

$$A + 2B = 13 \quad \text{and } 6A + 5B = 43.$$

Solving these two equations, we get $A = 3$ and $B = 5$

$$\therefore \frac{13x+43}{2x^2+17x+30} = \frac{3}{2x+5} + \frac{5}{x+6}.$$

7.2 Partial Fractions of $\frac{f(x)}{g(x)}$, when $g(x)$ contains repeated and / or non-repeated linear factors

To find the partial fractions of $\frac{f(x)}{g(x)}$ when $g(x)$ contains repeated linear factors, we use the following rule.

7.2.1 Rule II

Let $\frac{f(x)}{g(x)}$ *be a proper fraction. To each factor* $(ax+b)^n$, $a \neq 0$, *where* n *is a positive integer, of g(x) there will be partial fractions of the form* $\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots\cdots + \frac{A_n}{(ax+b)^n}$ *where* $A_1, A_2, ..., A_n$ *are the constants to be determined. Note that* $A_n \neq 0$ *and Rule I is a particular case of Rule II for n = 1.*

7.2.2 Solved Problems

1. Problem : *Resolve* $\frac{x^2+5x+7}{(x-3)^3}$ *into partial fractions.*

Solution : Let $\frac{x^2+5x+7}{(x-3)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3}$,

where A, B and C are constants to be determined.

$$\therefore \frac{x^2+5x+7}{(x-3)^3} = \frac{A(x-3)^2 + B(x-3) + C}{(x-3)^3}$$

$$\therefore x^2+5x+7 = Ax^2 + (B-6A)x + (9A-3B+C) \quad \text{... (1)}$$

Now, comparing the coefficients of like powers of x in (1), we get

$$A = 1, \quad B - 6A = 5, \quad 9A - 3B + C = 7$$

Solving these equations, we get $A = 1, \quad B = 11, \quad C = 31$.

$$\therefore \frac{x^2 + 5x + 7}{(x-3)^3} = \frac{1}{(x-3)} + \frac{11}{(x-3)^2} + \frac{31}{(x-3)^3}.$$

Another Method

Let $x - 3 = y$. Then $x = y + 3$

$$\therefore \frac{x^2 + 5x + 7}{(x-3)^3} = \frac{(y+3)^2 + 5(y+3) + 7}{y^3}$$

$$= \frac{y^2 + 11y + 31}{y^3}$$

$$= \frac{1}{y} + \frac{11}{y^2} + \frac{31}{y^3}$$

$$= \frac{1}{x-3} + \frac{11}{(x-3)^2} + \frac{31}{(x-3)^3}.$$

2. Problem : *Resolve* $\dfrac{x^2 + 13x + 15}{(2x+3)(x+3)^2}$ *into partial fractions.*

Solution : Here $(2x + 3)$ is a non-repeated linear factor and $(x + 3)$ is a repeated linear factor. We apply Rules I and II, and write

$$\frac{x^2 + 13x + 15}{(2x+3)(x+3)^2} = \frac{A}{2x+3} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

where A, B, C are constants to be determined.

$$\therefore A(x+3)^2 + B(2x+3)(x+3) + C(2x+3) = x^2 + 13x + 15. \quad \dots (1)$$

Putting $x = -3$ in (1), we get, $-3C = -15$ or $C = 5$.

Putting $x = \dfrac{-3}{2}$ in (1), we get, $\dfrac{9A}{4} = \dfrac{-9}{4}$ or $A = -1$.

Now comparing the coefficients of x^2 in (1), we get

$$A+2B = 1$$

i.e., $$-1+2B = 1 \qquad (\because A = -1)$$

$$B = 1$$

$$\therefore \frac{x^2+13x+15}{(2x+3)(x+3)^2} = \frac{-1}{2x+3} + \frac{1}{x+3} + \frac{5}{(x+3)^2}.$$

3. Problem : *Resolve* $\frac{1}{(x-1)^2(x-2)}$ *into partial fractions.*

Solution : Let $\frac{1}{(x-1)^2(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-2)}$

$$\therefore \frac{1}{(x-1)^2(x-2)} = \frac{A(x-1)(x-2)+B(x-2)+C(x-1)^2}{(x-1)^2(x-2)}$$

$$\therefore 1 = (A+C)x^2 + (-3A+B-2C)x + (2A-2B+C) \qquad \ldots (1)$$

Equating the corresponding coefficients, we have

$$A+C = 0, \quad -3A+B-2C = 0, \quad 2A-2B+C = 1$$

Solving these equations, we get

$$A = -1, \quad B = -1, \quad C = 1$$

$$\therefore \frac{1}{(x-1)^2(x-2)} = -\frac{1}{(x-1)} - \frac{1}{(x-1)^2} + \frac{1}{(x-2)}$$

4. Problem : *Resolve* $\frac{3x-18}{x^3(x+3)}$ *into partial fractions.*

Solution : Let $\frac{3x-18}{x^3(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+3}$

$$\therefore \frac{3x-18}{x^3(x+3)} = \frac{Ax^2(x+3)+Bx(x+3)+C(x+3)+Dx^3}{x^3(x+3)}$$

$$\therefore 3x-18 = (A+D)x^3 + (3A+B)x^2 + (3B+C)x + 3C \qquad \ldots (1)$$

Equating the corresponding coefficients, we have

$$A+D = 0, \quad 3A+B = 0, \quad 3B+C = 3, \quad 3C = -18$$

Solving these equations, we get

$$A = -1, \quad B = 3, \quad C = -6, \quad D = 1$$

$$\therefore \quad \frac{3x-18}{x^3(x+3)} = \frac{-1}{x} + \frac{3}{x^2} - \frac{6}{x^3} + \frac{1}{(x+3)}.$$

5. Problem : *Resolve* $\frac{x-1}{(x+1)(x-2)^2}$ *into partial fractions.*

Solution : Let $\frac{x-1}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$$\Rightarrow \quad x-1 = A(x-2)^2 + B(x+1)\cdot(x-2) + C(x+1) \qquad \text{... (1)}$$

Putting $x = 2$ in (1), we get $3C = 1$ i.e., $C = \frac{1}{3}$.

Putting $x = -1$ in (1), we get $9A = -2$ i.e., $A = -\frac{2}{9}$.

Putting $x = 0$ in (1), we get $4A - 2B + C = -1$ i.e., $B = \frac{2}{9}$.

$$\therefore \quad \frac{x-1}{(x+1)(x-2)^2} = \frac{-2}{9(x+1)} + \frac{2}{9(x-2)} + \frac{1}{3(x-2)^2}.$$

Exercise 7(a)

Resolve the following fractions into partial fractions.

I. 1. $\frac{2x+3}{(x+1)(x-3)}$ 2. $\frac{5x+6}{(2+x)(1-x)}$

II. 1. $\frac{3x+7}{x^2-3x+2}$ 2. $\frac{x+4}{(x^2-4)(x+1)}$ 3. $\frac{2x^2+2x+1}{x^3+x^2}$

4. $\frac{2x+3}{(x-1)^3}$ 5. $\frac{x^2-2x+6}{(x-2)^3}$

III. 1. $\frac{x^2-x+1}{(x+1)(x-1)^2}$ 2. $\frac{9}{(x-1)(x+2)^2}$ 3. $\frac{1}{(1-2x)^2(1-3x)}$

4. $\frac{1}{x^3(x+a)}$ 5. $\frac{x^2+5x+7}{(x-3)^3}$ 6. $\frac{3x^3-8x^2+10}{(x-1)^4}$

7.3 Partial Fractions of $\frac{f(x)}{g(x)}$, when $g(x)$ contains irrreducible quadratic factors

To find the partial fractions of $\frac{f(x)}{g(x)}$, when $g(x)$ contains *non-repeated* irreducible quadratic factors, we use the following rule.

7.3.1 Rule III

Let $\frac{f(x)}{g(x)}$ be a proper fraction. To each non repeated irreducible quadratic factor (ax^2+bx+c), $(a \neq 0)$ of $g(x)$ there will be a partial fraction of the form $\frac{Ax+B}{(ax^2+bx+c)}$ where A, B are real numbers, to be determined.

7.3.2 Solved Problems

1. Problem : *Resolve $\frac{2x^2+1}{x^3-1}$ into partial fractions.*

Solution :

$$\frac{2x^2+1}{x^3-1} = \frac{2x^2+1}{(x-1)(x^2+x+1)}$$

$$\therefore \text{ Let } \frac{2x^2+1}{(x-1)(x^2+x+1)} = \frac{\mathrm{A}}{x-1}+\frac{\mathrm{B}x+\mathrm{C}}{x^2+x+1}$$

$$= \frac{\mathrm{A}(x^2+x+1)+(\mathrm{B}x+\mathrm{C})(x-1)}{(x-1)(x^2+x+1)}$$

$$\therefore\ 2x^2+1 = (\mathrm{A}+\mathrm{B})x^2+(\mathrm{A}-\mathrm{B}+\mathrm{C})x+(\mathrm{A}-\mathrm{C})$$

$\therefore$ Comparing the corresponding coefficients, we have

$$\mathrm{A}+\mathrm{B} = 2, \quad \mathrm{A}-\mathrm{B}+\mathrm{C} = 0, \quad \mathrm{A}-\mathrm{C} = 1$$

Solving these equations, we get

$$\mathrm{A}=1, \quad \mathrm{B}=1, \quad \mathrm{C}=0$$

$$\therefore\ \frac{2x^2+1}{(x^3-1)} = \frac{1}{x-1}+\frac{x}{x^2+x+1}.$$

2. Problem : *Resolve* $\dfrac{x^3+x^2+1}{(x^2+2)\,(x^2+3)}$ *into partial fractions.*

Solution : Let $\dfrac{x^3+x^2+1}{(x^2+2)\,(x^2+3)} = \dfrac{Ax+B}{x^2+2} + \dfrac{Cx+D}{x^2+3}$

$$= \frac{(Ax+B)\,(x^2+3)+(Cx+D)(x^2+2)}{(x^2+2)\,(x^2+3)}$$

$\therefore\; x^3+x^2+1 = (A+C)x^3+(B+D)x^2+(3A+2C)\,x+(3B+2D)$

Comparing the corresponding coefficients, we have

$A+C = 1,\;\; B+D = 1,\;\; 3A+2C = 0,\;\; 3B+2D = 1$

Solving these equations, we get

$A = -2,\;\; B = -1,\;\; C = 3,\;\; D = 2$

$$\therefore\; \frac{x^3+x^2+1}{(x^2+2)\,(x^2+3)} = \frac{-2x-1}{(x^2+2)} + \frac{3x+2}{x^2+3} = \frac{(3x+2)}{(x^2+3)} - \frac{(2x+1)}{(x^2+2)}.$$

3. Problem : *Resolve* $\dfrac{3x^3-2x^2-1}{x^4+x^2+1}$ *into partial fractions.*

Solution : $x^4+x^2+1 = (x^2+1)^2 - x^2$

$= (x^2+x+1)(x^2-x+1)$

$$\therefore\; \text{Let } \frac{3x^3-2x^2-1}{x^4+x^2+1} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1}$$

$\Rightarrow (Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1) = 3x^3-2x^2-1$ ... (1)

Comparing the coefficients of x^3, x^2, x and constants in (1), we get

$A+C=3$ ... (2)

$-A+B+C+D = -2$ ... (3)

$A-B+C+D = 0$ ... (4)

$B+D = -1$ ... (5)

From (2), $C = 3-A$ ... (6)

and from (5), $D = -1-B$ ... (7)

Putting these values in (3) we get

$$-A + B + 3 - A - 1 - B = -2$$

$$\therefore\ -2A = -4 \text{ or } A = 2.$$

Similarly from (4) we get

$$A - B + 3 - A - 1 - B = 0$$

$$\therefore\ -2B = -2 \text{ or } B = 1$$

$$\therefore \text{ from (6): } C = 3 - A = 3 - 2 = 1$$

and from (7): $D = -1 - B = -1 - 1 = -2$

$$\therefore\ \frac{3x^3 - 2x^2 - 1}{x^4 + x^2 + 1} = \frac{2x+1}{x^2+x+1} + \frac{x-2}{x^2-x+1}.$$

To find the partial fractions of $\frac{f(x)}{g(x)}$, when $g(x)$ contains *repeated* irreducible quatradic factor, we use the following rule.

7.3.3 Rule IV

Let $\frac{f(x)}{g(x)}$ *be a proper fraction. If* $n(>1) \in \mathbf{N}$ *is the largest exponent so that* $(ax^2 + bx + c)^n, a \neq 0,$ *is a factor of g(x), then corresponding to each such factor, there will be partial fractions of the form*

$$\frac{A_1x + B_1}{(ax^2+bx+c)} + \frac{A_2x + B_2}{(ax^2+bx+c)^2} + \cdots\cdots + \frac{A_nx + B_n}{(ax^2+bx+c)^n}$$

where the real numbers A_i's and B_i's *are to be determined.*

We note that one of A_n *and* B_n *is different from zero.*

7.3.4 Solved Problems

1 Problem : *Resolve* $\frac{x^4 + 24x^2 + 28}{(x^2+1)^3}$ *into partial fractions.*

Solution : Let $\frac{x^4 + 24x^2 + 28}{(x^2+1)^3} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{(x^2+1)^3}$

$$= \frac{(Ax+B)(x^2+1)^2 + (Cx+D)(x^2+1) + (Ex+F)}{(x^2+1)^3}$$

$$\therefore\ x^4 + 24x^2 + 28 = Ax^5 + Bx^4 + (2A+C)x^3 + (2B+D)x^2 + (A+C+E)x + (B+D+F)$$

Comparing the corresponding coefficients, we have

$A = 0,\ B = 1,\ 2A + C = 0,\ 2B + D = 24,\ A + C + E = 0,\ B + D + F = 28.$

Solving these equations, we get

$A = 0,\quad B = 1,\quad C = 0,\quad D = 22,\quad E = 0,\quad F = 5$

$$\therefore \quad \frac{x^4 + 24x^2 + 28}{(x^2+1)^3} = \frac{1}{(x^2+1)} + \frac{22}{(x^2+1)^2} + \frac{5}{(x^2+1)^3}.$$

2. Problem : *Resolve* $\dfrac{x+3}{(1-x)^2(1+x^2)}$ *into partial fractions.*

Solution : Let $\dfrac{x+3}{(1-x)^2(1+x^2)} = \dfrac{A}{(1-x)} + \dfrac{B}{(1-x)^2} + \dfrac{Cx+D}{(1+x^2)}$

$\Rightarrow\ x + 3 = A(1-x)(1+x^2) + B(1+x^2) + (Cx + D)(1-x)^2$

Comparing the coefficients of like powers of x, we get

$$A + B + D = 3 \qquad \text{.... (1)}$$

$$-A + C - 2D = 1 \qquad \text{.... (2)}$$

$$A + B - 2C + D = 0 \qquad \text{.... (3)}$$

$$-A + C = 0 \qquad \text{.... (4)}$$

Solving these equations, we get

$$A = \frac{3}{2},\quad B = 2,\quad C = \frac{3}{2},\quad D = -\frac{1}{2}$$

$$\therefore \quad \frac{x+3}{(1-x)^2 + (1+x^2)} = \frac{3}{2(1-x)} + \frac{2}{(1-x)^2} + \frac{3x-1}{2(1+x^2)}$$

Exercise 7(b)

Resolve the following fractions into partial fractions.

1. $\dfrac{2x^2+3x+4}{(x-1)(x^2+2)}$

2. $\dfrac{3x-1}{(1-x+x^2)(x+2)}$

3. $\dfrac{x^2-3}{(x+2)\left(x^2+1\right)}$

4. $\dfrac{x^2+1}{\left(x^2+x+1\right)^2}$

5. $\dfrac{x^3+x^2+1}{(x-1)(x^3-1)}$

7.3.5 Partial fractions of $\frac{f(x)}{g(x)}$, when $\frac{f(x)}{g(x)}$ is an improper fraction

In finding the partial fractions of $\frac{f(x)}{g(x)}$, when $\frac{f(x)}{g(x)}$ is an improper fraction, the following two cases arise, which are illustrated with an example.

Case (i) : If the degree of $f(x)$ = the degree of $g(x)$ then by division algorithm there exists a unique constant k and polynomial $g(x)$ such that

$$f(x) = k.g(x) + r(x) \qquad \ldots (1)$$

where either $r(x) = 0$ or the degree of $r(x) <$ the degree of $g(x)$ and the constant k is the quotient of the coefficients of the highest degree terms of $f(x)$ and $g(x)$. Since $g(x) \neq 0$, from (1), $\frac{f(x)}{g(x)}$ can be expressed as $\frac{f(x)}{g(x)} = k + \frac{r(x)}{g(x)}$ where $\frac{r(x)}{g(x)}$ is a proper fraction which can be resolved into partial fractions using the rules studied in 7.1.1, 7.2.1, 7.3.1 and 7.3.3.

7.3.6 Solved Problem

Problem : *Resolve* $\frac{x^3}{(2x-1)(x+2)(x-3)}$ *into partial fractions.*

Solution : The given fraction is improper with degree of numerator equal to degree of denominator. Note that $x^3 = \frac{1}{2}(2x-1)(x+2)(x-3) + r(x)$ where degree of $r(x)$ is less than $3(=\text{degree of } (2x-1)(x+2)(x-3))$, and hence

$$\frac{x^3}{(2x-1)(x+2)(x-3)} = \frac{\frac{1}{2}(2x-1)(x+2)(x-3) + r(x)}{(2x-1)(x+2)(x-3)}$$

$$= \frac{1}{2} + \frac{r(x)}{(2x-1)(x+2)(x-3)}$$

$$\therefore \text{ Let } \frac{x^3}{(2x-1)(x+2)(x-3)} = \frac{1}{2} + \frac{A}{(2x-1)} + \frac{B}{(x+2)} + \frac{C}{(x-3)}.$$

$$\therefore\ (2x-1)(x+2)(x-3)+2A(x+2)(x-3)$$
$$+2B(2x-1)(x-3)+2C(2x-1)(x+2)=2x^3 \quad \ldots (1)$$

Now taking $x=\frac{1}{2}$ in (1) we get

$$2A\,.\left(\frac{1}{2}+2\right)\left(\frac{1}{2}-3\right) = 2\left(\frac{1}{2}\right)^3$$

$$\therefore\ \frac{-50}{4}A = \frac{1}{4} \quad \text{or,} \quad A = \frac{-1}{50}$$

taking $x=-2$ in (1) we get $50B=-16$ or, $B=\frac{-8}{25}$

taking $x=3$ in (1) we get $50C=54$ or, $C=\frac{27}{25}$

$$\therefore\ \frac{x^3}{(2x-1)(x+2)(x-3)} = \frac{1}{2}-\frac{1}{50(2x-1)}-\frac{8}{25(x+2)}+\frac{27}{25(x-3)}.$$

Case (ii) : If $\frac{f(x)}{g(x)}$ is an improper fraction with degree of $f(x) >$ the degree of $g(x)$, then by using division algorithm 7.0.5, it can be expressed as $\frac{f(x)}{g(x)} = q(x)+\frac{r(x)}{g(x)}$ where $q(x)$ is a non-zero polynomial and $\frac{r(x)}{g(x)}$ is a proper fraction. Further $\frac{r(x)}{g(x)}$ can be resolved into partial fractions using the rules in 7.1.1, 7.2.1, 7.3.1 and 7.3.3. A method of finding partial fractions of an improper fraction when degree of $f(x) >$ the degree of $g(x)$, is given in the following illustration.

7.3.7 Solved Problem

Problem : *Resolve* $\frac{x^4}{(x-1)(x-2)}$ *into partial fractions.*

Solution : By dividing x^4 with $(x-1)(x-2)$ we get

$$\frac{x^4}{(x-1)(x-2)} = x^2+3x+7+\frac{15x-14}{x^2-3x+2}$$

Now the partial fraction decomposition of

$$\frac{15x-14}{x^2-3x+2} \text{ is } \frac{-1}{x-1}+\frac{16}{x-2}$$

Hence $$\frac{x^4}{(x-1)(x-2)} = x^2+3x+7-\frac{1}{x-1}+\frac{16}{x-2}.$$

Exercise 7(c)

Resolve the following fractions into partial fractions.

1. $\dfrac{x^2}{(x-1)(x-2)}$

2. $\dfrac{x^3}{(x-1)(x+2)}$

3. $\dfrac{x^3}{(2x-1)(x-1)^2}$

4. $\dfrac{x^3}{(x-a)(x-b)(x-c)}$

7.3.8 Conversion of $\dfrac{f(x)}{g(x)}$ in power series of x

When $|x|<1$, $\dfrac{1}{1-x}$, $\dfrac{1}{1+x}$, $\dfrac{1}{(1-x)^2}$ and $\dfrac{1}{(1+x)^2}$ have the "Power series expansions"

$$\frac{1}{1-x} = 1+x+x^2+\cdots\cdots+x^k+\cdots\cdots$$

$$\frac{1}{1+x} = 1-x+x^2-\cdots\cdots+(-1)^k x^k+\cdots\cdots$$

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\cdots\cdots+(k+1)x^k+\cdots\cdots$$

$$\frac{1}{(1+x)^2} = 1-2x+3x^2-4x^3+\cdots\cdots+(-1)^k(k+1)x^k+\cdots\cdots,$$

and $$\frac{1}{(1-x)^n} = 1+nx+\frac{n(n+1)}{2}x^2+\cdots\cdots \text{ where } n \text{ is an integer.}$$

These expansions can be used to express some proper fractions as power series of x, as given in the succeeding illustrations.

7.3.9 Solved Problems

1. Problem : *Find the coefficient of* x^4 *in the expansion of* $\dfrac{3x}{(x-2)(x+1)}$ *in powers of* x *specifying the interval in which the expansion is valid.*

Solution : Resolving the given fraction into partial fractions we get

$$\frac{3x}{(x-2)(x+1)} = \frac{2}{(x-2)} + \frac{1}{(x+1)}$$

$$\text{Now, } \frac{2}{(x-2)} + \frac{1}{(x+1)} = \frac{2}{(-2)\left(1-\frac{x}{2}\right)} + \frac{1}{(1+x)}$$

$$= -\left(1-\frac{x}{2}\right)^{-1} + (1+x)^{-1} \qquad \dots (1)$$

$$\text{Now } \left(1-\frac{x}{2}\right)^{-1} = 1+\left(\frac{x}{2}\right)+\left(\frac{x}{2}\right)^2+\cdots\cdots+\left(\frac{x}{2}\right)^n+\cdots\cdots,\ \text{if } \left|\frac{x}{2}\right| < 1, \text{ i.e.,} |x| < 2$$

$$\text{and } (1+x)^{-1} = 1-x+x^2-\cdots\cdots+(-1)^n x^n+\cdots\cdots,\ \text{ if } |x|<1$$

Thus if $|x|<1$ both the above expansions are valid and hence we get, from (1),

$$\frac{2}{x-2}+\frac{1}{x+1} = -\left[1+\left(\frac{x}{2}\right)+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\left(\frac{x}{2}\right)^4+\cdots\cdots\right] + \ 1-x+x^2-x^3+x^4-\cdots\cdots$$

$$\therefore \text{ Coefficient of } x^4 = -\frac{1}{16}+1 = \frac{15}{16}.$$

2. Problem: *Find the coefficient of* x^n *in the power series expansion of* $\dfrac{x}{(x-1)^2(x-2)}$ *specifying the region in which the expansion is valid.*

Solution : Resolving the given fraction into partial fractions, we get

$$\frac{x}{(x-1)^2(x-2)} = \frac{-2}{(x-1)} - \frac{1}{(x-1)^2} + \frac{2}{(x-2)}$$

$$= \frac{2}{(1-x)} - \frac{1}{(1-x)^2} + \frac{2}{(-2)(1-\frac{x}{2})}$$

$$= 2(1-x)^{-1} - (1-x)^{-2} - (1-\frac{x}{2})^{-1}$$

$$= 2\left[1+x+x^2+\cdots\cdots\right]-\left[1+2x+3x^2+4x^3+\cdots\cdots\right]$$

$$-\left[1+\left(\frac{x}{2}\right)+\left(\frac{x}{2}\right)^2+\cdots\cdots\right], \text{ if } |x|<1$$

$\therefore$ Coefficient of x^n in this expansion is

$$= 2-(n+1)-\left(\frac{1}{2}\right)^n$$

$$= 1-n-\left(\frac{1}{2}\right)^n.$$

Exercise 7(d)

1. Find the coefficient of x^3 in the power series expansion of $\dfrac{5x+6}{(x+2)(1-x)}$ specifying the region in which the expansion is valid.

2. Find the coefficient of x^4 in the power series expansion of $\dfrac{3x^2+2x}{(x^2+2)(x-3)}$ specifying the interval in which the expansion is valid.

3. Find the coefficient of x^n in the power series expansion of $\dfrac{x-4}{x^2-5x+6}$ specifying the region in which the expansion is valid.

4. Find the coefficient of x^n in the power series expansion of $\dfrac{3x}{(x-1)(x-2)^2}$.

Key Concepts

Sl. No.	Factor of $g(x)$	The form of the partial fractions of $\frac{f(x)}{g(x)}$ corresponding to the factor of $g(x)$
1.	$(ax+b)$	$\frac{A}{(ax+b)}$
2.	$(ax+b)^n$, $a \neq 0$, and $n\,(>1) \in \mathbf{N}$	$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$
3.	Irreducible ax^2+bx+c, $a \neq 0$,	$\frac{Ax+B}{(ax^2+bx+c)}$
4.	$(ax^2+bx+c)^n$, $a \neq 0$, and $n\,(>1) \in \mathbf{N}$	$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$

Historical Note

The Method of Partial Fractions was introduced by *Johann Bernoulli* (1667-1748), who was instrumental in the early development of Calculus. He was a professor of the university of Basel in Switzerland and taught higher mathematics to *Leonhard Euler*.

Answers

Exercise 7(a)

I. 1. $\frac{9}{4(x-3)} - \frac{1}{4(x+1)}$ 2. $\frac{11}{3(1-x)} - \frac{4}{3(2+x)}$

II. 1. $\frac{-10}{x-1} + \frac{13}{x-2}$ 2. $\frac{1}{2(x-2)} + \frac{1}{2(x+2)} - \frac{1}{x+1}$ 3. $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$

4. $\frac{2}{(x-1)^2} + \frac{5}{(x-1)^3}$ 5. $\frac{1}{(x-2)} + \frac{2}{(x-2)^3}$

III. 1. $\frac{3}{4(x+1)}+\frac{1}{4(x-1)}+\frac{1}{2(x-1)^2}$

2. $\frac{1}{x-1}-\frac{1}{x+2}-\frac{3}{(x+2)^2}$

3. $\frac{9}{1-3x}-\frac{6}{1-2x}-\frac{2}{(1-2x)^2}$

4. $\frac{1}{a^3x}-\frac{1}{a^2x^2}+\frac{1}{ax^3}-\frac{1}{a^3(x+a)}$

5. $\frac{1}{x-3}+\frac{11}{(x-3)^2}+\frac{31}{(x-3)^3}$

6. $\frac{3}{(x-1)}+\frac{1}{(x-1)^2}-\frac{7}{(x-1)^3}+\frac{5}{(x-1)^4}$

Exercise 7(b)

1. $\frac{3}{x-1}+\frac{2-x}{x^2+2}$

2. $\frac{x}{1-x+x^2}-\frac{1}{x+2}$

3. $\frac{1}{5(x+2)}+\frac{4x-8}{5(x^2+1)}$

4. $\frac{1}{x^2+x+1}-\frac{x}{(x^2+x+1)^2}$

5. $\frac{2}{3(x-1)}+\frac{1}{(x-1)^2}+\frac{x+2}{3(x^2+x+1)}$

Exercise 7(c)

1. $1-\frac{1}{x-1}+\frac{4}{x-2}$

2. $x-1+\frac{1}{3(x-1)}+\frac{8}{3(x+2)}$

3. $\frac{1}{2}+\frac{1}{2(2x-1)}+\frac{1}{(x-1)}+\frac{1}{(x-1)^2}$

4. $1+\frac{a^3}{(a-b)(a-c)(x-a)}+\frac{b^3}{(b-c)(b-a)(x-b)}+\frac{c^3}{(c-a)(c-b)(x-c)}$

Exercise 7(d)

1. $\frac{15}{4}$

2. $\frac{77}{324}$

3. $\frac{1}{3^{n+1}}-\frac{1}{2^n}$

4. $-3+\frac{3}{2^{n+1}}+\frac{3(n+1)}{2^{n+1}}$

Probability

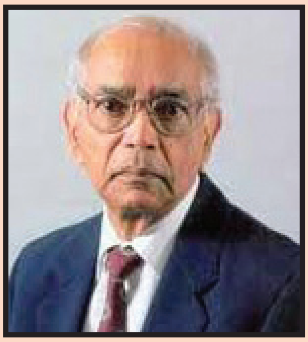
Chapter 8

Measures of Dispersion

"Scientific laws are not advanced by the principle of authority or justified by faith or midieval philosophy; statistics is the only court of appeal to new knowledge" ***- P.C. Mahalanobis***

Introduction

In the earlier classes you have studied some methods of representing data graphically and pictorially. This representation reveals certain salient features or characteristics of the data. We have learnt how to construct a frequency table for a given data and find various measures that provide a single representative value of the data, called a measure of central tendency. Recall that arithmetic mean, median, mode, geometric mean and harmonic mean are the measures of central tendency. A measure of central tendency gives us a rough idea where data points are centered. But this single representative value cannot adequately describe the variation of a set of data. For example the mean of the set of values 2, 10, 87 is 33. The mean of the distribution 30, 32, 37 is also 33. So, to make better representation of the data, we should have an idea how the data is spread or scattered or dispersed or how much they are bunched around a measure of central tendency.



C.R. Rao

(10th Sep. 1920 -)

Prof. C. R. Rao was born in India on 10th September 1920. He was graduated from Andhra University, India and Kings college, Cambridge University, U.K. He was a Fellow of the Royal society (U.K.) and was the Director of the Indian Statistical Institute, Kolkota.

The Government of India honoured him with the second highest civilian award, 'Padma Vibhushan' for his outstanding contribution to Statistics. Recently Prof. C.R. Rao has been honoured establishment of an Advanced Institute of Mathematics, Statistics and Computer Science (AIMSCS), named after him for doing basic research in the University of Hyderabad Campus, Hyderabad.

A measure of dispersion or variation describes the spread or scattering of the individual values around the central value. To illustrate the concept of dispersion, we shall consider the following example.

Consider the following data of runs scored by two batsmen A, B in the last ten test matches.

A	28	70	30	2	42	64	80	93	5	116
B	44	38	60	45	52	54	49	54	76	58

Note that the arithmetic mean $\left(\frac{\Sigma x_i}{N}\right)$ of the scores of both the players is 53 and the median score ($\frac{n+1}{2}$, if n is odd and $\frac{1}{2}\left(\frac{n}{2}+\frac{n+2}{2}\right)$, if n is even) is also 53. Based on these values of measures of central tendency, can we say that the performance of these two batsmen is the same? The answer is clearly 'no' because the variability of the scores of A is from 2 to 116 whereas the variability of the runs scored by B is between 38 and 76. This measure 'variability' is another factor required to be studied in Statistics. Like a measure of central tendency, we have to know a measure to describe the variability. This measure is called the 'measure of dispersion'.

Measuring dispersion of a data is significant because it determines the reliability of an average by pointing out as to how far an average is representative of the entire data. In this chapter, we shall learn the following measures of dispersion and their methods of calculation for ungrouped and grouped data.

(i) Range (ii) Mean deviation (iii) Standard deviation

such a measure computed for a distribution is called 'Statistic'.

8.1 Range

For an ungrouped data, range is defined as the difference between the maximum (greatest) value and the minimum (smallest) value of the series of observations.

For a grouped data (i.e., data given in the form of a frequency table), range is approximated as the difference between the upper limit of the largest class and the lower limit of the smallest class.

1. Example : In the example of runs scored by the two batsmen A and B, we can make some inference on the variability in the scores on the basis of minimum and maximum runs in each series. This difference is called the range of the data.

In the case of batsman A, the range is $116 - 2 = 114$ and for batsman B, the range is $76 - 38 = 38$. Clearly, the range of A is greater than the range of B. Therefore, the scores are more scattered or dispersed in the case of A, where as for B, they are less dispersed and close to each other.

2. Example : Let us consider the daily sales (in Rs.) of two firms A and B, for 5 days, as given in the following table :

Firm A	Firm B
5,050	4,900
5,025	3,100
4,950	2,200
4,835	1,800
5,140	13,000
$\overline{X}_A = 5{,}000$	$\overline{X}_B = 5{,}000$

The average sales of both firms is the same but the distribution pattern of the sales is not similar. There is a greater amount of variation in the daily sales of the firm B than that of the firm A.

Range of sales of firm A $= 5140 - 4835 = 305$.

Range of sales of firm B $= 13000 - 1800 = 11{,}200$.

The range of a data is very easy to calculate and it gives us some idea about the variability of the data. However, the range is a crude measure of dispersion, since it uses only two extreme values. Also, it does not tell us about the dispersion of the data from a measure of central tendency. Hence we need to know a more realistic measure of dispersion known as (i) mean deviation and (ii) standard deviation.

8.2 Mean Deviation

To find the dispersion of values of x from a central value 'a', we find the deviations about 'a'. They are $(x - a)$'s. To find the mean deviation we have to sum up all such deviations. Since a measure of central tendency lies between the maximum and minimum values of a distribution, some of the deviations will be negative and some positive. Also, some of the deviations may vanish. If, in particular $a = \overline{x}$, the sum of the deviations from the mean ($\overline{x}$) is zero. In this case,

$$\text{Mean of the deviations from the arithmetic mean} = \frac{\text{Sum of the deviations}}{\text{Number of observations}} = \frac{0}{n} = 0.$$

Hence finding such mean deviation does not serve any purpose.

Recall that the absolute value of the difference of two numbers gives the distance between the numbers when represented on a number line. Hence, to find the measure of dispersion from a fixed number 'a', we may take the mean of the absolute values of the deviations from the arithmetic mean ($\overline{x}$). Such mean is called the mean deviation from the arithmetic mean and expressed as

$$\text{Mean deviation from the mean} = \frac{\text{Sum of the absolute values of deviations from } \overline{x}}{\text{Number of observations}}.$$

Remark : Mean deviation may be obtained from any measure of central tendency. For instance, it can be obtained from median also. Mean deviation from mean and median are commonly used.

We shall now learn how to compute the mean deviation from mean and median.

8.2.1(a) Mean Deviation from the mean for ungrouped data

Suppose we have a discrete data with n observations $x_1, x_2, ..., x_n$. Then we adopt the following procedure for computing the mean deviation from the mean of the given data.

Step 1: Calculate the arithmetic mean ($\bar{x}$) of the n observations. Let it be 'a'.

Step 2: Find the deviations of each x_i from 'a', i.e., $x_1 - a, x_2 - a, ..., x_n - a$.

Step 3: Find the absolute value i.e., $|x_1 - a|, |x_2 - a|, ... , |x_n - a|$ of these deviations by ignoring the negative sign, if any, in the deviations computed in step 2.

Step 4: Find the arithmetic mean of the absolute values of the deviations.

$$\text{i.e., M.D from the mean} = \frac{\sum_{i=1}^{n} |x_i - a|}{n}.$$

3. Example : Find the mean deviation from the mean of the following discrete data :

6, 7, 10, 12, 13, 4, 12, 16.

Solution : The arithmetic mean of the given data is

$$\bar{x} = \frac{6+7+10+12+13+4+12+16}{8} = 10.$$

The absolute values of the deviations : $|x_i - \bar{x}|$ are 4, 3, 0, 2, 3, 6, 2 , 6.

$$\therefore \text{ The mean deviation from the mean } = \frac{\sum_{i=1}^{8} |x_i - \bar{x}|}{8}$$

$$= \frac{4+3+0+2+3+6+2+6}{8} = \frac{26}{8} = 3.25.$$

8.2.1(b) Mean Deviation from the median for an ungrouped data

Step 1 : Calculate the median of the n observations $x_1, x_2, ..., x_n$. Let it be 'b'.

Step 2 : Find the deviation of each x_i from b i.e., $x_1 - b, x_2 - b, ..., x_n - b$.

Step 3 : Find the absolute values of these deviations i.e., $|x_1 - b|, |x_2 - b|, ..., |x_n - b|$.

Step 4 : Find their arithmetic mean as M.D. from median $= \dfrac{\sum_{i=1}^{n} |x_i - b|}{n}$.

4. Example : Compute the mean deviation about the median of the data given in Example 1.

Solution : Expressing the data points in the ascending order of magnitude, we get 4, 6, 7, 10, 12, 12, 13, 16.

Then the median (b) of these 8 observations is $\dfrac{10+12}{2} = 11$.

Then the absolute values $|x_i - b|$ are 7, 5, 4, 1, 1, 1, 2, 5

$\therefore$ Mean deviation from the median $= \dfrac{\sum_{i=1}^{8} |x_i - b|}{8} = \dfrac{26}{8} = 3.25$.

Note : (i) In the problem we considered above, the obtained mean deviation from the mean and the mean deviation from the median are equal. But in general they need not be equal.

(ii) The value of the mean deviation about the median of an ungrouped data is the least when compared to the mean deviations computed about any other measure of central tendency. This is also called the minimal property of the median.

8.2.2 Mean Deviation for a grouped data

You have learnt in elementary statistics that a data can be arranged or grouped as a frequency distribution in two ways : (i) Discrete frequency distribution and (ii) Continuous frequency distribution.

We shall now discuss the method of finding the mean deviation for both the types of distributions.

(a) **Discrete frequency distribution :** Suppose the data consists of n distinct points x_1, x_2, ..., x_n occurring with frequencies $f_1, f_2, ..., f_n$ respectively. Then, we can represent this data in the following manner :

x_i	x_1	x_2	x_3	.....	x_n
f_i	f_1	f_2	f_3	.....	f_n

This form is called the discrete frequency distribution.

8.2.2(i) Mean Deviation about the mean

Recall that the arithmetic mean ($\bar{x}$) of a discrete frequency distribution with n data points is obtained

using the formula : $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i} = \dfrac{1}{N} \sum_{i=1}^{n} x_i f_i,$

where $N = \sum_{i=1}^{n} f_i$ gives the total frequencies in the considered distribution.

Now the mean deviation about the mean i.e., M.D. ($\bar{x}$) is obtained by finding the absolute values of the deviations of the data points from the mean i.e., $|x_i - \bar{x}|$ and using the formula :

$$\text{M.D(mean)} = \frac{\sum_{i=1}^{n} f_i |x_i - \bar{x}|}{\sum_{i=1}^{n} f_i} = \frac{1}{N} \sum_{i=1}^{n} f_i |x_i - \bar{x}|.$$

5. Example : Let us find the mean deviation about the mean for the following data

x_i	2	5	7	8	10	35
f_i	6	8	10	6	8	2

We shall now construct the following table to enable us to compute the required statistic.

x_i	f_i	$f_i\, x_i$	$\lvert x_i - \bar{x}\rvert$	$f_i \lvert x_i - \bar{x}\rvert$
2	6	12	6	36
5	8	40	3	24
7	10	70	1	10
8	6	48	0	0
10	8	80	2	16
35	2	70	27	54
	$N=\Sigma f_i = 40$	$\Sigma f_i\, x_i=320$		140

$$\text{A.M.} \quad \bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{320}{40} = 8.$$

$$\text{M.D. (mean)} = \frac{1}{N} \sum_{i=1}^{6} f_i |x_i - \bar{x}| = \frac{1}{40} \times 140 = 3.5.$$

8.2.2(ii) Mean Deviation about the median

To find the mean deviation about the median, we have to find the median of the given discrete frequency distribution. After arranging the observations in either ascending or in descending order, we shall then find the sum of the frequencies : $\Sigma f_i = \mathrm{N}$ and compute the cumulative frequencies. Then we shall identify the observation whose cumulative frequency is equal to or just greater than N/2. This is the median of the data.

To obtain the mean deviation about the median, we find the absolute values of the deviations from the median and substitute them in the formula :

$$\text{M.D. (median)} = \frac{1}{\mathrm{N}} \sum_{i=1}^{n} f_i \,|\, x_i - \text{median} \,|$$

6. Example : Find the mean deviation from the median for the following data.

x_i	6	9	3	12	15	13	21	22
f_i	4	5	3	2	5	4	4	3

Here $\Sigma f_i = 30$

Solution : Keeping the observations in the ascending order, we get the following distribution:

x_i	3	6	9	12	13	15	21	22
f_i	3	4	5	2	4	5	4	3

Median of these observations is the mean of the 15^{th} and 16^{th} observations, which is equal to 13.

Now we compute the absolute values of the deviations i.e., $|x_i - \text{med.}|$, from the median and compute $f_i|x_i - \text{med.}|$, as shown in the following table :

$\lvert x_i - \text{med.}\rvert$	10	7	4	1	0	2	8	9
f_i	3	4	5	2	4	5	4	3
$f_i \lvert x_i - \text{med.}\rvert$	30	28	20	2	0	10	32	27

Now, $\sum_{i=1}^{8} f_i \,|\, x_i - \text{med.} | = 149.$

Hence mean deviation from the median

$$= \frac{1}{\mathrm{N}} \sum_{i=1}^{8} f_i \,|\, x_i - \text{med.} | = \frac{1}{30} \times 149 = 4.97.$$

8.2.3(i) Finding mean deviation from the mean for a continuous frequency distribution

Recall that a continuous frequency distribution is a series in which the data is classified into different class-intervals (without gaps) along with their respective frequencies (f_i). As an illustration, consider the following grouped data in the form of continuous distribution which relate to the sales of 100 companies :

Sales (in Rs.thousand)	40-50	50-60	60-70	70-80	80-90	90-100
Number of companies	5	15	25	30	20	5

Recall that while computing the arithmetic mean of a continuous frequency distribution, we assumed that the entire frequency f_i of the *i*-th class interval is centred at the midpoint x_i of that class interval. In the discussion that follows, we adopt in much the same procedure and write the midpoint (x_i) of each class interval. With these x_i, we proceed to find the mean deviation, as has been done in the case of a discrete frequency distribution.

We shall now illustrate the computation of mean deviation with the sales data of 100 companies given above.

7. Example : Mean deviation from the mean : We shall construct the following table from the given tabular data :

Sales (in Rs. thousand)	Number of companies f_i	Mid point of calss interval x_i	$f_i x_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
40 - 50	5	45	225	26	130
50 - 60	15	55	825	16	240
60 - 70	25	65	1625	6	150
70 - 80	30	75	2250	4	120
80 - 90	20	85	1700	14	280
90 - 100	5	95	475	24	120
	$\Sigma f_i = N = 100$		$\Sigma f_i x_i = 7100$		$\Sigma f_i \lvert x_i - \bar{x} \rvert = 1040$

Here $N = \Sigma f_i = 100$ and $\bar{x} = \frac{\Sigma f_i x_i}{N} = \frac{7100}{100} = 71$.

Hence mean deviation from the mean $= \frac{1}{N} \Sigma f_i \, | x_i - \bar{x} |$

$$= \frac{1}{100}(1040) = 10.4 \cdot$$

Alternative simple method (Step - deviation method)

Some times when the mid points of the class intervals x_i as well as their associated frequencies are numerically large, then we find a computational tediousness in the above procedure. To avoid this tediousness, we take an assumed mean 'a' which lies in the middle or just close to it in the data and take the deviations of mid points x_i from this assumed mean. This amounts to shifting of origin from zero to the assumed mean on the number line.

Some times, if there is a common factor of all the deviations, we divide them by this common factor (h) to further simplify the deviations. These are known as step deviations. Taking step deviations amounts to change of scale on the number line.

With the assumed mean 'a' and common factor 'h', if we define a new variable $d_i = \frac{x_i - a}{h}$, then the arithmetic mean $\bar{x} = a + \left(\left(\sum_{i=1}^{n} f_i d_i \right) \Big/ N \right) h$.

(Remark: When $d_i = \frac{x_i - a}{h}$, then $x_i = a + h\, d_i$. Multiplying throughout by f_i, taking summation on both sides from 1 to n and dividing throughout by N we get $\bar{x} = a + \left(\left(\sum_{i=1}^{n} f_i d_i \right) \Big/ N \right) h$.

We shall now illustrate this simplified procedure with the following example :

8. Example : Find the mean deviation about the mean for the following data :

Marks obtained	0-10	10-20	20-30	30-40	40-50
No.of students	5	8	15	16	6

Solution : Taking the assumed mean $a = 25$ and $h = 10$, we form the following table.

Class interval	frequency f_i	Mid point x_i	$d_i = \frac{x_i - 25}{10}$	$f_i d_i$	$\lvert x_i - \overline{x} \rvert$	$f_i \lvert x_i - \overline{x} \rvert$
0 - 10	5	5	−2	−10	22	110
10 - 20	8	15	−1	−8	12	96
20 - 30	15	25	0	0	2	30
30 - 40	16	35	1	16	8	128
40 - 50	6	45	2	12	18	108
	$\Sigma f_i = 50$			$\Sigma f_i d_i = 10$		$\Sigma f_i \lvert x_i - \overline{x} \rvert = 472$

Now, $\overline{x} = a + \left(\left(\sum_{i=1}^{5} f_i d_i \right) \Big/ \mathrm{N} \right) h = 25 + \left(\frac{10}{50} \right) 10 = 27$.

Hence mean deviation from the mean $= \frac{1}{\mathrm{N}} \sum_{i=1}^{5} f_i \lvert x_i - \overline{x} \rvert = \frac{1}{50}(472) = 9.44$ marks.

8.2.3(ii) Mean Deviation from the median

The process of finding the mean deviation from the median for a continuous frequency distribution is just similar to the procedure adopted for finding the mean deviation about the mean. The only difference lies in the replacement of mean by median while taking the deviations.

To find the median for a continuous frequency distribution, we identify the class interval in which $\mathrm{N}/2$ th observation lies. This class is known as the 'median class'. We then find the median, using the formula:

$$\text{Median} = \mathrm{L} + \left[\left(\frac{\mathrm{N}}{2} - p.c.f \right) \Big/ f \right] i, \text{ where}$$

L is the lower limit of the median class, $p.c.f$ is the preceding cumulative frequency to the median class, f is the frequency of the median class and i is the width of the median class.

After finding the median, the absolute values of the deviations of the mid point x_i of each class from the median i.e., $|x_i - \text{median}|$ are found.

Then mean deviation from median $= \frac{1}{N}\sum_{i=1}^{n} f_i \, |x_i - \text{med.}|$

This process is illustrated with the following data relating to the age distribution of workers in an industrial establishment.

9. Example

Age (years)	20-25	25-30	30-35	35-40	40-45	45-50	50-55	55-60
No.of workers (f_i)	120	125	175	160	150	140	100	30

Solution : We form the following table for the given data :

Class interval	frequency f_i	Cumulative frequency c.f.	Mid point x_i	$\|x_i - \text{med.}\|$	$f_i\|x_i - \text{med.}\|$
20-25	120	120	22.5	15	1800
25-30	125	245	27.5	10	1250
30-35	175	420	32.5	5	875
35-40	160	580	37.5	0	0
40-45	150	730	42.5	5	750
45-50	140	870	47.5	10	1400
50-55	100	970	52.5	15	1500
55-60	30	1000	57.5	20	600
	N=1000				$\sum_{i=1}^{8} f_i \, \|x_i - \text{med.}\| = 8175$

Here $\frac{N}{2}^{\text{th}}$ observation $= \frac{1000}{2} = 500^{\text{th}}$ observation lies in the class interval 35 - 40. This is the median class.

$$\therefore \text{ Median } = L + \{[(N/2) - p.c.f]/f\}\, i = 35 + \left(\frac{500-420}{160}\right) \times 5$$

$$= 35 + \frac{400}{160} = 35 + 2.5 = 37.50 .$$

$$\therefore \text{ Mean deviation about median } = \frac{1}{N}\sum_{i=1}^{8} f_i\,|x_i - \text{med.}|$$

$$= \frac{1}{1000} \times 8175 = 8.175 .$$

8.3 Variance and Standard Deviation of ungrouped / grouped data

In the earlier section, while finding the mean deviation about the mean or median, we have taken the absolute values of the deviations to give meaning to that statistic, otherwise the deviations may cancel among themselves. To overcome this difficulty that arise due to the signs of deviations, we consider the squares of the deviations to make them non-negative. Thus if $x_1, x_2, ..., x_n$ are n observations and $\bar{x}$ is their mean, then

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + ... + (x_n - \bar{x})^2 \geq 0 .$$

We have the following cases :

Case (i) : If $\sum_{i=1}^{n}(x_i - \bar{x})^2 = 0$, then each $(x_i - \bar{x}) = 0$ which implies all observations are equal to the mean $\bar{x}$ and hence there is no dispersion.

Case(ii) : If $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is small, then it indicates that each observation x_i is very close to the mean $\bar{x}$ and hence the degree of dispersion is low.

Case (iii) : If $\sum_{i=1}^{n}(x_i - \bar{x})^2$ is large, then it indicates a higher degree of dispersion of the observations from the mean $\bar{x}$.

On the otherhand, if we take the mean of the squared deviations from the mean, i.e., $\frac{1}{n}\Sigma(x_i - \bar{x})^2$, then it is found that this number leads to a proper measure of dispersion. This number is called 'variance' and is denoted by σ^2(read as sigma square) and is given by $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2$. Then σ, the standard deviation is given by the positive square root of the variance.

$$\therefore \quad \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2} \, .$$

8.3.1 (i) Calculation of variance and standard deviation for an ungrouped data

10. Example : Find the variance and standard deviation of the following data :

5, 12, 3, 18, 6, 8, 2, 10.

Solution : The mean of the given data is

$$\overline{x} = \frac{\Sigma x_i}{n} = \frac{5+12+3+18+6+8+2+10}{8} = \frac{64}{8} = 8 \, .$$

To find the variance, we construct the following table :

x_i	5	12	3	18	6	8	2	10
$x_i - \overline{x}$	−3	4	−5	10	−2	0	−6	2
$(x_i - \overline{x})^2$	9	16	25	100	4	0	36	4

Here $\Sigma(x_i - \overline{x})^2 = 194$.

$$\therefore \text{ variance } (\sigma^2) = \frac{1}{n}\sum_{i=1}^{8}(x_i - \overline{x})^2 = \frac{1}{8}\times 194 = 24.25 \, .$$

Hence standard deviation $(\sigma) = \sqrt{24.45} = 4.95$(approx).

8.3.1(ii) Calculation of variance and standard deviation for a discrete frequency distribution.

11. Example :

x_i	4	8	11	17	20	24	32
f_i	3	5	9	5	4	3	1

We shall construct the following table for computing the required statistic :

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
4	3	12	−10	100	300
8	5	40	−6	36	180
11	9	99	−3	9	81
17	5	85	3	9	45
20	4	80	6	36	144
24	3	72	10	100	300
32	1	32	18	324	324
		$\Sigma f_i x_i = 420$			$\Sigma f_i(x_i - \bar{x})^2 = 1374$

Here $N = 30, \sum_{i=1}^{7} f_i x_i = 420, \sum_{i=1}^{7} f_i (x_i - \bar{x})^2 = 1374$.

$$\bar{x} = \frac{\Sigma f_i x_i}{N} = \frac{420}{30} = 14.$$

$$\text{Variance } (\sigma^2) = \frac{1}{N} \Sigma f_i (x_i - \bar{x})^2 = \frac{1}{30} \times 1374 = 45.8.$$

$$\text{Standard deviation } (\sigma) = \sqrt{45.8} = 6.77.$$

8.3.2 Standard Deviation of a continuous frequency distribution

Recall that in the case of finding mean deviation for a continuous frequency distribution, we have transformed it as a discrete distribution by representing each class by its mid point. Then the variance standard deviation is calculated by adopting the same procedure that was carried out for a discrete frequency distribution.

If there are n classes in a given distribution, each class represented by its mid point x_i with frequency f_i, the standard deviation is obtained using the formula

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \bar{x})^2}, \text{ where } N = \sum_{i=1}^{n} f_i \text{ and } \bar{x} \text{ is the mean of the distribution.}$$

Aliter : For the purpose of simplifying the computation in finding the standard deviation, we adopt the following alternative formula.

We have, variance (σ^2) $= \dfrac{1}{N}\sum_{i=1}^{n} f_i (x_i - \overline{x})^2$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i (x_i^2 + \overline{x}^2 - 2x_i\overline{x})$$

$$= \frac{1}{N}\left[\sum_{i=1}^{n} f_i\, x_i^2 + \sum_{i=1}^{n} \overline{x}^2\, f_i - \sum_{i=1}^{n} 2\overline{x}\;\; f_i x_i\right]$$

$$= \frac{1}{N}\left[\sum_{i=1}^{n} f_i\, x_i^2 + \overline{x}^2 \sum_{i=1}^{n} f_i - 2\overline{x}\sum_{i=1}^{n} f_i x_i\right]$$

$$= \frac{1}{N}\left[\sum_{i=1}^{n} f_i\, x_i^2 + \overline{x}^2 N - 2\overline{x}.N\overline{x}\right], \qquad \left(\because \frac{1}{N}\sum x_i\, f_i = \overline{x} \text{ or } \Sigma f_i x_i = N\overline{x}\right)$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i\, x_i^2 + \overline{x}^2 - 2\overline{x}^2$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i\, x_i^2 - \overline{x}^2 \text{ or } \frac{1}{N}\sum_{i=1}^{n} f_i\, x_i^2 - \left(\frac{\Sigma f_i x_i}{N}\right)^2$$

Then, standard deviation $(\sigma) = \sqrt{\dfrac{1}{N}\Sigma f_i\, x_i^2 - \left(\dfrac{\Sigma f_i x_i}{N}\right)^2}$

Some times the mid points x_i of the different class intervals in a continuous distribution are so large that the calculation of mean and variance becomes tedious and time consuming. In such cases, we apply the step deviation method, as detailed here under, to avoid the complexity in computation.

Let h be the width of the class intervals and A be the assumed mean. Assume that the scale is reduced to $1/h$ times in the step deviation method.

Define $y_i = \dfrac{x_i - A}{h}$, $i = 1, 2, ..., n.$

Then $x_i = A + h\, y_i$ ...(1)

Now $\bar{x} = \dfrac{\sum_{i=1}^{n} f_i\, x_i}{N} = \dfrac{\sum_{i=1}^{n} f_i\, (A + hy_i)}{N}$, by (1)

$$= \frac{1}{N}\left[\sum_{i=1}^{n} A\, f_i + \sum_{i=1}^{n} h\, f_i y_i\right] = \frac{1}{N}\left[A\sum_{i=1}^{n} f_i + h\sum_{i=1}^{n} f_i y_i\right]$$

$$= A + h\left[\sum_{i=1}^{n} f_i y_i / h\right] = A + h\,\bar{y} \qquad ...(2)$$

Also $\sigma_x^2 = \dfrac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x})^2$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i (A + hy_i - A - h\bar{y})^2, \text{ using (1) and (2)}$$

$$= \frac{1}{N}\sum_{i=1}^{n} f_i h^2 (y_i - \bar{y})^2 = h^2\left[\frac{1}{N}\sum_{i=1}^{n} f_i (y_i - \bar{y})^2\right]$$

$$= h^2\, \sigma_y^2 \text{ or } \sigma_x = h\,\sigma_y \qquad ...(3)$$

But we have shown that S.D $(\sigma_x) = \dfrac{1}{N}\sqrt{N\,\Sigma f_i x_i^2 - (\Sigma f_i\, x_i)^2}$

or $\sigma_y = \dfrac{1}{N}\sqrt{N\,\Sigma f_i y_i^2 - (\Sigma f_i\, y_i)^2}$

Hence from equation (3), $\sigma_x = \dfrac{h}{N}\sqrt{N\sum_{i=1}^{n} f_i y_i^2 - \left(\sum_{i=1}^{n} f_i y_i\right)^2}$

We shall exemplify this simple method with the following example.

12. Example : Calculate the variance and standard deviation of the following continuous frequency distribution.

Class interval	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

Solution : Here $h = 10$. If we take the assumed mean A = 65, then $y_i = \dfrac{x_i - 65}{10}$. We shall now construct the following table with the given data.

Class interval (C.I.)	Frequency (f_i)	Mid Point of C.I. (x_i)	$y_i = \dfrac{x_i - 65}{10}$	y_i^2	$f_i\, y_i$	$f_i y_i^2$
30-40	3	35	−3	9	−9	27
40-50	7	45	−2	4	−14	28
50-60	12	55	−1	1	−12	12
60-70	15	65	0	0	0	0
70-80	8	75	1	1	8	8
80-90	3	85	2	4	6	12
90-100	2	95	3	9	6	18
	N = 50				$\Sigma f_i y_i = -15$	$\Sigma\, f_i y_i^2 = 105$

Mean $\quad \bar{x} = A + \left(\dfrac{\Sigma\, f_i y_i}{N}\right) \times h = 65 - \left(\dfrac{15}{50} \times 10\right) = 62$.

Variance $\quad (\sigma_x^2) = \dfrac{h^2}{N^2}\left[N\,\Sigma\, f_i y_i^2 - (\Sigma f_i\, y_i)^2\right]$

$$= \frac{100}{2500}\left[50(105) - (-15)^2\right] = \frac{1}{25}[5250 - 225] = 201.$$

Standard deviation $\sigma_x = \sqrt{201} = 14.18$.

8.4 Coefficient of Variation and analysis of frequency distributions with equal means but different variances

The two measures of dispersion we have studied in this unit namely, mean deviation and standard deviation, have the same units in which the data is given. Whenever we want to compare the variability of two series of data having the same mean (or may differ widely in their mean) or measured in different units, we do not merely calculate the measures of dispersion. Instead, we require a measure which is independent of units. The measure of variability which is a pure number and is independent of units is called the coefficient of variation, denoted by C.V.

The coefficient of variation of a distribution is defined as C.V. = $\frac{\sigma}{\bar{x}} \times 100, \ \bar{x} \neq 0$,

where σ is the standard deviation and $\bar{x}$ is the mean of the distribution or data.

The coefficient of variation is a relative measure of variation. For comparing the variability of two series (distributions), we calculate the coefficient of variation, for each series. The series having greater C.V. is said to have more variability than the other. The series having less C.V. is deemed to be more consistant (or homogeneous) than the other.

8.4.1 Analysis / Comparison of two frequency distributions with equal means

Suppose we have two distributions D_1 and D_2 with the same mean i.e., $\bar{x}_1 = \bar{x}_2 = \bar{x}$ (say), but different standard deviations; σ_1 and σ_2 respectively. Then C.V. of $D_1 = (\sigma_1 / \bar{x}) \times 100$ and C.V of $D_2 = (\sigma_2 / \bar{x}) \times 100$. Then it follows that the two C.V.'s can be compared on the basis of the values of σ_1 and σ_2 only. In this case, the series with lower value of standard deviation is said to be more consistant than the other and the series with greater standard deviation is called more dispersed than the other.

13. Example : Students of two sections A and B of a class show the following performance in a test (conducted for 100 marks)

	Section A	Section B
Number of students	50	60
Average marks in the test	45	45
Variance of distribution of marks	64	81

Which section of students has greater variability in performance?

Solution : Since variance of distribution of marks of section A is 64, its standard deviation $\sigma_1 = 8$. Similarly, since the variance of distribution of marks of section B is 81, its standard deviation $\sigma_2 = 9$.

Since the average marks of both sections of students is the same i.e., 45, the section with greater standard deviation will have more variability. Hence section B has greater variability in the performance.

8.4.2 Comparison of two distributions with unequal means

We shall now illustrate this case by considering an example :

14. Example : Lives of two models of refrigerators A and B, obtained in a survey, are given below :

Life (in years)	Model A	Model B
0 - 2	5	2
2 - 4	16	7
4 - 6	13	12
6 - 8	7	19
8 - 10	5	9

Which refrigerator model would you suggest to purchase ?

Solution : To find the mean and variance of the lives of Model A and Model B of refrigerators, we shall construct the following table.

Class interval	Mid Point	Model A			Model B		
	x_i	f_i	$f_i x_i$	$f_i x_i^2$	f_i	$f_i x_i$	$f_i x_i^2$
0 - 2	1	5	5	5	2	2	2
2 - 4	3	16	48	144	7	21	63
4 - 6	5	13	65	325	12	60	300
6 - 8	7	7	49	343	19	133	931
8 - 10	9	5	45	405	9	81	729
		N = 46	212	1221	N = 49	297	2025

Model A : $\bar{x}_A = \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \dfrac{212}{46} = 4.6.$

Model B : $\bar{x}_B = \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \dfrac{297}{49} = 6.06.$

Now $\sigma_A^2 = \dfrac{1221}{46} - \left(\dfrac{212}{46}\right)^2 = 5.38 \qquad \Rightarrow \sigma_A = \sqrt{5.38} = 2.319.$

$\sigma_B^2 = \dfrac{2025}{49} - \left(\dfrac{297}{49}\right)^2 = 4.61 \qquad \Rightarrow \sigma_B = \sqrt{4.61} = 2.147.$

Coefficient of variation of model A $= \dfrac{\sigma_A}{\bar{x}_A} \times 100 = \dfrac{2.319}{4.6} \times 100 = 50.41$

Coefficient of variation of model B $= \dfrac{\sigma_B}{\bar{x}_B} \times 100 = \dfrac{2.147}{6.06} \times 100 = 35.43.$

Since C.V. of model B < C.V. of model A, we can say that model B is more consistent than the model A, with regard to life in years. Hence we suggest model B for purchase.

8.5 Solved Problems

1. Problem : *Find the mean deviation from the mean of the following data, using the step deviation method.*

Marks	0-10	10-20	20-30	30-40	40-50	50-60	60-70
No.of students	6	5	8	15	7	6	3

Solution : To find the required statistic, we shall construct the following table:

Class interval	Mid point (x_i)	Number of students (f_i)	$d_i = \frac{x_i - 35}{10}$	$f_i d_i$	$\lvert x_i - \bar{x} \rvert = \lvert x_i - 33.4 \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
0 - 10	5	6	−3	−18	28.4	170.4
10 - 20	15	5	−2	−10	18.4	92
20 - 30	25	8	−1	−8	8.4	67.2
30 - 40	35	15	0	0	1.6	24.0
40 - 50	45	7	1	7	11.6	81.2
50 - 60	55	6	2	12	21.6	129.6
60 - 70	65	3	3	9	31.6	94.8
		N = 50		$\Sigma f_i d_i = -8$		659.2

Here N = 50. Mean ($\bar{x}$) = $A + \frac{h(\Sigma f_i d_i)}{N}$

$$= 35 + \frac{10(-8)}{50} = 33.4 \text{ marks.}$$

Mean Deviation from mean $= \frac{1}{N}\Sigma f_i \lvert x_i - \bar{x}_i \rvert = \frac{1}{50}(659.2)$

$= 13.18$ (nearly).

2. Problem : *The following table gives the daily wages of workers in a factory. Compute the standard deviation and the coefficient of variation of the wages of the workers.*

Wages (Rs.)	125-175	175-225	225-275	275-325	325-375	375-425	425-475	475-525	525-575
Number of workers	2	22	19	14	3	4	6	1	1

Solution : We shall solve this problem using the step deviation method, since the mid points of the class intervals are numerically large.

Here $h = 50$. Take $a = 300$. Then $y_i = \frac{x_i - 300}{50}$.

Mid point of C.I x_i	frequency f_i	y_i	$f_i y_i$	$f_i y_i^2$
150	2	−3	−6	18
200	22	−2	−44	88
250	19	−1	−19	19
300	14	0	0	0
350	3	1	3	3
400	4	2	8	16
450	6	3	18	54
500	1	4	4	16
550	1	5	5	25
	N = 72		$\Sigma f_i y_i = -31$	$\Sigma f_i y_i^2 = 239$

Mean $\bar{x} = A + \left(\frac{\Sigma f_i y_i}{N}\right) \times h = 300 + \left(\frac{-31}{72}\right)50 = 300 - \frac{1550}{72} = 278.47.$

$$\text{Variance} \quad (\sigma_x^2) = \frac{h^2}{N^2}\left[N\sum f_i y_i^2 - (\Sigma f_i y_i)^2\right]$$

$$= \frac{2500}{72 \times 72}[72(239) - (31 \times 31)]$$

$$\sigma_x = \sqrt{2500\left(\frac{239}{72} - \frac{961}{72 \times 72}\right)} = 88.52.$$

Coefficient of variation $= \frac{88.52}{278.47} \times 100 = 31.79$.

3. Problem : *An analysis of monthly wages paid to the workers of two firms A and B belonging to the same industry gives the following data*

	Firm A	*Firm B*
Number of workers	*500*	*600*
Average daily wage (Rs.)	*186*	*175*
Variance of distribution of wages	*81*	*100*

(i) *Which firm A or B, has greater variability in individual wages ?*

(ii) *Which firm has larger wage bill ?*

Solution : Since variance of distribution of wages in firm A is 81, $\sigma_1^2 = 81$ and hence $\sigma_1 = 9$.

Since variance of distribution of wages in firm B is 100, $\sigma_2^2 = 100$ and hence $\sigma_2 = 10$.

$$\therefore \text{ C.V. of distribution of wages of firm A} = \frac{\sigma_1}{\bar{x}_1} \times 100 = \frac{9}{186} \times 100 = 4.84.$$

$$\text{C.V. of distribution of wages of firm B} = \frac{\sigma_2}{\bar{x}_2} \times 100 = \frac{10}{175} \times 100 = 5.71.$$

Since C.V. of firm B is greater than C.V. of firm A, we can say that firm B has greater variability in individual wages.

(ii) Firm A has number of workers i.e., wage earners $(n_1) = 500$.

Its average daily wage, say $\bar{x}_1$ = Rs.186.

Since Average daily wage $= \dfrac{\text{Total wages paid}}{\text{No. of workers}}$, it follows that total wages paid to the workers

$$= n_1 \, \bar{x}_1 = 500 \times 186 = \text{Rs.}93{,}000.$$

Firm B has number of wage earners $(n_2) = 600$.

Average daily wage, say $\bar{x}_2$ = Rs.175.

$\therefore$ Total daily wages paid to the workers $= n_2 \bar{x}_2 = 600 \times 175 =$ Rs.1,05,000.

Hence we see that firm B has larger wage bill.

4. Problem : *The variance of 20 observations is 5. If each of the observations is multiplied by 2, find the variance of the resulting observations.*

Solution : Let the given observations be $x_1, x_2, ..., x_{20}$ and $\bar{x}$ be their mean.

Given that $n = 20$ and variance = 5

$$\text{i.e.,} \quad \frac{1}{20}\sum_{i=1}^{20}(x_i - \bar{x})^2 = 5 \quad \text{or} \quad \sum_{i=1}^{20}(x_i - \bar{x})^2 = 100. \qquad \text{... (1)}$$

If each observation is multiplied by 2, then the new observations are

$$y_i = 2x_i, \; i = 1, 2, ..., 20 \;\text{ or }\; x_i = y_i / 2.$$

Therefore $\bar{y} = \frac{1}{20}\sum_{i=1}^{20} y_i = \frac{1}{20}\sum_{i=1}^{20} 2x_i = 2.\frac{1}{20}\sum_{i=1}^{20} x_i = 2\bar{x}$ or $\bar{x} = \frac{1}{2}\bar{y}$

Substituting the values of x_i and $\bar{x}$ in (1) we get

$$\sum_{i=1}^{20}\left(\frac{1}{2}y_i - \frac{1}{2}\bar{y}\right)^2 = 100 \;\text{ i.e., }\; \sum_{i=1}^{20}\left(y_i - \bar{y}\right)^2 = 400.$$

Then the variance of the resulting observations $= \frac{1}{20} \times 400 = 20 = 2^2 \times 5.$

Note : From this example we note that, if each observation in a data is multiplied by a constant k, then the variance of the resulting observations is k^2 times that of the variance of original observations.

5. Problem : *If each of the observations $x_1, x_2, ..., x_n$ is increased by k, where k is a positive or negative number, then show that the variance remains unchanged.*

Solution : Let $\bar{x}$ be the mean of $x_1, x_2, ..., x_n$. Then their variance is given by $\sigma_1^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$.

If to each observation we add a constant k, then the new (changed) observations will be

$$y_i = x_i + k \qquad \ldots (1)$$

Then the mean of the new observations $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i + k) = \frac{1}{n}\left[\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} k\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i + \frac{1}{n}(nk) = \overline{x} + k \qquad \ldots (2)$$

The variance of the new observations $= \sigma_2^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \overline{y})^2$

$$= \frac{1}{n}\sum_{i=1}^{n}(x_i + k - \overline{x} - k)^2, \quad \text{using (1) and (2)}$$

$$= \frac{1}{n}(x_i - \overline{x})^2 = \sigma_1^2.$$

Thus the variance of the new observations is the same as that of the original observations.

Note : We note that adding (or subtracting) a positive number to (or from) each of the given set of observations does not affect the variance.

6. Problem : *The scores of two cricketers A and B in 10 innings are given below. Find who is a better run getter and who is a more consistent player.*

Scores of A : x_i	40	25	19	80	38	8	67	121	66	76
Scores of B : y_i	28	70	31	0	14	111	66	31	25	4

Solution : For cricketer A : $\overline{x} = \frac{540}{10} = 54$; For cricketer B : $\overline{y} = \frac{380}{10} = 38$.

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	y_i	$y_i - \bar{y}$	$(y_i - \bar{y})^2$
40	−14	196	28	−10	100
25	29	841	70	32	1024
19	−35	1225	31	−7	49
80	26	676	0	−38	1444
38	−16	256	14	−24	575
8	−46	2116	111	73	5329
67	13	169	66	28	784
121	67	4489	31	−7	49
66	12	144	25	−13	163
76	22	484	4	−34	1156
$\Sigma x_i = 540$		10596	$\Sigma y_i = 380$		10680

Standard deviation of scores of A $= \sigma_x = \sqrt{\frac{1}{n}\Sigma(x_i - \bar{x})^2} = \sqrt{\frac{10596}{10}} = \sqrt{1059.6} = 32.55$.

Standard deviation of the scores of B $= \sigma_y = \sqrt{\frac{1}{n}\Sigma(y_i - \bar{y})^2} = \sqrt{\frac{10680}{10}} = \sqrt{1068} = 32.68$.

C.V of A $= \frac{\sigma_x}{\bar{x}} \times 100 = \frac{32.55}{54} \times 100 = 60.28.$

C.V. of B $= \frac{\sigma_y}{\bar{y}} \times 100 = \frac{32.68}{38} \times 100 = 86.$

Since $\bar{x} > \bar{y}$, cricketer A is a better run getter (scorer). Since C.V. of A < C.V. of B, cricketer A is also a more consistent player.

Exercise 8(a)

I. 1. Find the mean deviation about the mean for the following data

(i) 38, 70, 48, 40, 42, 55, 63, 46, 54, 44

(ii) 3, 6, 10, 4, 9, 10

2. Find the mean deviation about the median for the following data.

(i) 13, 17, 16, 11, 13, 10, 16, 11, 18, 12, 17

(ii) 4, 6, 9, 3, 10, 13, 2

3. Find the mean deviation about the mean for the following distribution.

(i)

x_i	10	11	12	13
f_i	3	12	18	12

(ii)

x_i	10	30	50	70	90
f_i	4	24	28	16	8

4. Find the mean deviation about the median for the following frequency distribution.

x_i	5	7	9	10	12	15
f_i	8	6	2	2	2	6

II. 1. Find the mean deviation about the median for the following continuous distribution.

(i)

Marks obtained	0-10	10-20	20-30	30-40	40-50	50-60
No.of boys	6	8	14	16	4	2

(ii)

Class interval	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
frequency	5	8	7	12	28	20	10	10

2. Find the mean deviation about the mean for the following continuous distribution.

Height (in cms)	95-105	105-115	115-125	125-135	135-145	145-155
Number of boys	9	13	26	30	12	10

3. Find the variance for the discrete data given below.

(i) 6, 7, 10, 12, 13, 4, 8, 12

(ii) 350, 361, 370, 373, 376, 379, 385, 387, 394, 395

4. Find the variance and standard deviation of the following frequency distribution

x_i	6	10	14	18	24	28	30
f_i	2	4	7	12	8	4	3

III. 1. Find the mean and variance using the step deviation method, of the following tabular data, giving the age distribution of 542 members.

Age in years (x_i)	20-30	30-40	40-50	50-60	60-70	70-80	80-90
Number of (f_i) members	3	61	132	153	140	51	2

2. The coefficient of variation of two distributions are 60 and 70 and their standard deviations are 21 and 16 respectively. Find their arithmetic means.

3. From the prices of shares X and Y given below, for 10 days of trading, find out which share is more stable ?

X	35	54	52	53	56	58	52	50	51	49
Y	108	107	105	105	106	107	104	103	104	101

4. The mean of 5 observations is 4.4. Their variance is 8.24. If three of the observations are 1, 2 and 6. Find the other two observations.

5. The arithmetic mean and standard deviation of a set of 9 items are 43 and 5 respectively. If an item of value 63 is added to that set, find the new mean and standard deviation of 10 item set given.

Key Concepts

- Measures of Dispersion :

  Range, mean deviation, variance, standard deviation are some measures of dispersion.

- Range is defined as the difference of maximum value and the minimum value of the data.

- Mean Deviation for ungrouped distribution.

  (i) Mean Deviation about the mean $= \frac{1}{n}\Sigma|x_i - \bar{x}|$.

  (ii) Mean Deviation about median $= \frac{1}{n}\Sigma|x_i - \text{Median}|$.

- Mean Deviation for grouped data

  (i) Mean deviation about mean $= \frac{1}{\text{N}}\Sigma f_i|x_i - \bar{x}|$, where $\text{N} = \Sigma f_i$.

  (ii) Mean Deviation about median $= \frac{1}{\text{N}}\Sigma f_i|x_i - \text{Median}|$, where $\text{N} = \Sigma f_i$

- Variance and standard deviation for ungrouped data

$$\sigma^2 = \frac{1}{n}\Sigma(x_i - \bar{x})^2,\ \sigma = \sqrt{\frac{1}{n}\Sigma(x_i - \bar{x})^2}$$

- Variance and standard deviation of a discrete frequency distribution.

$$\sigma^2 = \frac{1}{\text{N}}\Sigma f_i(x_i - \bar{x})^2,\quad \sigma = \sqrt{\frac{1}{\text{N}}\Sigma f_i(x_i - \bar{x})^2}$$

- Standard deviation of a continuous frequency distribution

$$\sigma = \frac{1}{\text{N}}\sqrt{\text{N}\Sigma f_i x_i^2 - (\Sigma f_i x_i)^2}$$

(or) $$\sigma = \frac{h}{\text{N}}\sqrt{\text{N}\Sigma f_i y_i^2 - (\Sigma f_i y_i)^2} \quad \text{where} \quad y_i = \frac{(x_i - \text{A})}{h}.$$

- Coefficient of variation $= \frac{\sigma}{\bar{x}} \times 100;\quad \bar{x} \neq 0.$

Historical Note

Statistics is one of the ancient branches of Science. It is as old as the human civilisation itself. Statistics is derived from the Latin word 'Status' which means a certain political state. In India, about 2000 years ago, during the regime of Chandra Gupta Mourya (324-300 B.C) historians have evidence of the existence of an efficient system of collecting administrative statistics. Kautilya's 'Arth Shastra' (300 B.C) made a specific mention of census data related to births and deaths. Abul Fazl, the author of 'Ain-I-Akbari' also described a detailed account of the various surveys conducted during the regime of Akbar.

Answers

Exercise 8(a)

I. 1. (i) 8.4
(ii) 2.67

2. (i) 2.45
(ii) 3.29

3. (i) 0.71
(ii) 16

4. 3.23

II. 1. (i) 10.35
(ii) 14.29

2. 11.28

3. (i) 9.25
(ii) 183.2

4. Variance = 43.4; Standard deviation = $\sqrt{43.4}$

III. 1. Mean = 54.72 years
Variance = 141.07 years

2. $\bar{x}_1 = 35$, $\bar{x}_2 = 22.85$

3. Y

4. 4 and 9

5. New Mean = 45
New Standard deviation = 7.65.

Chapter 9

Probability

" The theory of probability is simply the science of logic quantitatively treated"

- C.S. Peirce

Introduction

Probability is an important branch of mathematics that deals with the phenomenon of chance or randomness. In daily life, we talk informally about the possibility of some event to happen. For example, while leaving the house in the morning on a cloudy day, one may have to decide to take an umbrella even if it is not raining because it may possibly rain later in the day. The theory of probability is developed initially to explain such type of decisions mathematically. In other words, probability is a measure of uncertainity.

Historically, the theory of probability has its origin to gambling and games of chance. Chevalier de Mere (1607-1685), a French gambler approached a French mathematician Blaise Pascal (1623-1662) to find a solution to a problem related to gambling. After giving a solution to

Kolmogorov

(1903 - 1987)

Andrey Nikolaevich Kolmogorov *was a Soviet mathematician who made major advances in different scientific fields (among them are probability theory, topology, intuitionistic logic, turbulence, classical mechanics and computational complexity). Kolmogorov is widely considered as one of the prominent mathematicians of the 20^{th} century.*

Chevalier's problem, Pascal made correspondence with another French mathematician Fermat (1601-1665) to lay the foundations of the theory of probability.

The first attempt towards giving some mathematical rigour to this subject was done by the French mathematician, astronomer and physicist Laplace (1749-1827). In his monumental work "Theorie analytique des probabilities", Laplace gave the classical definition for the probability of an event.

The theory of probability, as we learn today is due to the contribution of Andrei Nikolaevich Kolmogorov (1903-1987) a Russian topologist and probabilist. He laid the set theoretic foundation to probability in his classic work "foundations of the theory of Probability" in 1933.

Although probability theory was initiated in the field of gambling, it now plays an essential role in several branches of science and engineering. It is extensively used in the study of genetics to help understand the inheritance of traits. In Computer Science, probability theory plays an important role in the study of the complexity of algorithms. This extensive application makes it an important branch of study.

In this chapter, we shall restrict our study to random experiments that result in a finitely many, equally likely outcomes. We shall define probability in a classical way and also through the axiomatic approach. Then we introduce some key concepts like equally likely, mutually exclusive and independent events and conditional probability. We shall state and prove the addition theorem, the multiplication theorem and the Bayes theorem and illustrate their applications through some examples.

9.1 Random Experiments and Events

Theory of probability is a study of random or unpredictable nature of experiments. It is helpful in investigating the important features of these experiments. A preliminary knowledge of set theory and permutations and combinations is a pre-requisite for the study of this topic. We shall first introduce some key concepts and terminology that is necessary to develop the theory.

9.1.1 Definition

An experiment that can be repeated any number of times under identical conditions in which:

(i) *All possible outcomes of the experiment are known in advance,*

(ii) *The actual outcome in a particular case is not known in advance,*
is called a random experiment.

We shall now give some examples of random experiments.

9.1.2 Examples

1. Example : In an experiment of tossing an unbiased coin, we have only two possible outcomes : Head (H) and Tail (T). In a particular trial, one does not know in advance, the outcome. This experiment can be performed any number of times under identical conditions. Therefore it is a random experiment.

2. Example : Rolling a fair die is also a random experiment. If we denote the six faces of the cubic die with the numbers 1, 2, 3, 4, 5 and 6, then the possible outcome of the experiment is one of the numbers 1, 2, 3, 4, 5 or 6 appearing on the uppermost face of the die. The six faces of a die may also contain dots in numbers 1, 2, 3, 4, 5, 6. In any case, we shall identify the faces of a die, hereafter with the numbers 1, 2, 3, 4, 5, 6.

3. Example : Tossing a fair coin till a tail appears is also a random experiment.

The experiments such as (i) measuring the acceleration due to gravity using a compound pendulum and (ii) measuring the volume of a gas by increasing the pressure, keeping the temperature fixed, are not random experiments.

In the discussions that follow in chapters 9 and 10, by a coin and a die, we mean always an unbiased (or fair) coin and unbiased (or fair) die unless specified otherwise.

9.1.3 Definitions

(i) *Any possible outcome of a random experiment $\mathscr{E}$ is called an elementary or simple event.*

(ii) *The set of all elementary events (possible outcomes) of a random experiment $\mathscr{E}$ is called the sample space S, associated with $\mathscr{E}$.*

That is, S is the sample space of a random experiment $\mathscr{E}$ if (a) every element of S is an outcome of $\mathscr{E}$ and (b) every performance of $\mathscr{E}$ results in an outcome that corresponds to exactly one element of S.

(iii) *An elementary event is a point of the sample space.*

(iv) *A subset* E *of* S, *is called an event.*

(v) *An event* E *is said to happen (or occur) if an outcome of the experiment belongs to* E. *Otherwise, we say that E has not happened (or not occurred).*

(vi) *The complement of an event* E, *denoted by* E^c, *is the event given by* $E^c = S - E$, *which is called the complementary event of* E.

(vii) *The empty set ϕ and the set* S, *being trivial subsets of* S, *are events called impossible event and certain (definite) event respectively.*

9.1.4 Examples

1. Example : In the experiment of 'tossing a coin', the sample space S is given by S = { H,T }, where H stands for head and T stands for tail. Occurrence of H and occurrence of T are the only two elementary (simple) events, while occurrence of either H or T is a certain (definite) event.

2. Example : In the experiment of rolling a die, the sample space S = { 1, 2, 3, 4, 5, 6 }. Occurrence of a number ≤ 6 on the uppermost face of the die is a certain (definite) event, whereas occurrence of an even number on the uppermost face of the die is an event.

9.1.5 Remark

Note that the sample space S of a random experiment $\mathscr{E}$ may or may not be finite. Throughout the chapters 9 and 10, S is taken to be either finite or countably infinite (A set S is said to be countably infinite if there exists a bijection from S to N, the set of natural numbers. If S is countably infinite, then S can be written as $S = \{s_1, s_2, ..., s_n, ...\}$. The set **Z** of integers, the set **Q** of rational numbers are some examples of countably infinite sets.)

Observe that the sample space S of the experiment given in examples 1 and 2 of 9.1.2 are finite, while that of example 3 is countably infinite.

9.1.6 Definitions

(i) *Two or more events are said to be* ***mutually exclusive*** *if the occurrence of one of the events prevents the occurrence of any of the remaining events. Thus events* $E_1, E_2, ..., E_k$ *are said to be mutually exclusive if* $E_i \cap E_j = \phi$ for $i \neq j,\ 1 \leq i,\ j \leq k$

(ii) *Two or more events are said to be* ***equally likely*** *(or equiprobable) if there is no reason to expect any one of them to happen in preference to the others.*

(iii) *Two or more events are said to be* ***exhaustive*** *if the performance of the experiment always results in the occurrence of atleast one of them.*

Thus events $E_1, E_2, ..., E_k$ *are said to be exhaustive if* $E_1 \cup E_2 \cup ..... \cup E_k = S$

The following examples illustrate these concepts :

9.1.7 Examples

1. Example : In the experiment of throwing a die, the event

E_1 : Occurrence of an even number (on the face of the die) and

E_2 : Occurrence of an odd number are mutually exclusive events. They are also exhaustive.

2. Example: In the experiment of throwing a pair of dice, let us consider the following events:

E_1 : A sum 7 (of the numbers that appear on the uppermost faces of the dice)
E_2 : A sum 6 (of the numbers that appear on the uppermost faces of the dice)
E_3 : A sum 5 (of the numbers that appear on the uppermost faces of the dice) and
E_4 : A sum $\geq$ 7 (of the numbers that appear on the uppermost faces of the dice)

Observe that the events E_1, E_2, E_3 are mutually exclusive, while events E_1 and E_4 are not mutually exclusive.

3. Example : If a coin is tossed, occurrence of a Head (H) or occurrence of a Tail (T) are equally likely.

4. Example : A pair of dice is thrown. The following table shows the outcome of sum of the numbers that appear on the two dice.

Table 1 : Sum of the numbers that shows up on the two dice.

		Die - 1					
		1	2	3	4	5	6
Die - 2	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Observe from the table, that the sum 6 occurs 5 times whereas the sum 3 occurs only twice. Hence the events : "Occurrence of the sum 6" and "Occurrence of the sum 3" are not equally likely events.

5. Example : In the experiment of rolling a die, the events :

E_1 : Occurrence of an even number
E_2 : Occurrence of a prime number, and
E_3 : Occurrence of 1

on the face that shows up are exhaustive, but not mutually exclusive events.

Exercise 9(a)

I. 1. In the experiment of throwing a die, consider the following events :
$A = \{1, 3, 5\},\ B = \{2, 4, 6\},\ C = \{1, 2, 3\}$. Are these events equally likely?

2. In the experiment of throwing a die, consider the following events :
$A = \{1, 3, 5\},\ B = \{2, 4\},\ C = \{6\}$. Are these events mutually exclusive?

3. In the experiment of throwing a die, consider the events
$A = \{2, 4, 6\},\ B = \{3, 6\},\ C = \{1, 5, 6\}$. Are these events exhaustive?

II. 1. Give two examples of mutually exclusive and exhaustive events.

2. Give examples of two events that are neither mutually exclusive nor exhaustive?

3. Give two examples of events that are neither equally likely nor exhaustive.

9.2 Classical definition of probability, Axiomatic approach and addition theorem of probability

In this section, we shall start with the classical definition of probability given by Laplace. Next we give the statistical (emperical) definition. We shall then present the Kolmogorov's axiomatic approach to probability which overcomes the short comings of the earlier versions. Lastly we shall state and prove the addition theorem of probability.

9.2.1 Classical (or Mathematical) definition of probability

If a random experiment results in n exhaustive, mutually exclusive and equally likely elementary events and m of them are favourable to the happening of an event E, then the probability of occurrence of E (or simply probability of E) denoted by P(E) is defined by

$$P(E) = \frac{m}{n}$$

From this definition it is clear that, for any event E, $0 \le P(E) \le 1$

Since the number of elementary events or outcomes not favourable to this event is $(n - m)$, the probability of non-occurrence of the event E, denoted by $P(E^c)$, is given by

$$P(E^c) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$

$$\therefore P(E) + P(E^c) = 1$$

9.2.2 Examples

1. Example : Two dice are thrown. We now find the probability of getting the same number on both the faces.

Solution : Let E be the event of getting the same number on both the faces of the two dice. So, the number of cases favourable to $E = 6$. The total number of elementary events or the points of the sample space = $6 \times 6 = 36$

Hence $P(E) = \frac{6}{36} = \frac{1}{6}$

2. Example : An integer is picked from 1 to 20, both inclusive. Let us find the probability that it is a prime.

Solution : The sample space S consists of 20 points. Let E be the event that the number picked is a prime. Then $E = \{2, 3, 5, 7, 11, 13, 17, 19\}$. Then the number of elements in E is 8. Hence the required probability is $P(E) = \frac{8}{20} = \frac{2}{5}$

3. Example : A bag contains 4 red, 5 black and 6 blue balls. Let us find the probability that two balls drawn at random simultaneously from the bag are a red and a black ball.

Solution : Total number of balls in the bag $= 4 + 5 + 6 = 15$. From out of these balls, two balls can be drawn in ${}^{15}C_2 = \frac{15 \times 14}{2} = 105$ ways.

Out of 4 red balls, one ball can be drawn in ${}^{4}C_1 = 4$ ways and out of 5 black balls, one ball can be drawn in ${}^{5}C_1 = 5$ ways.

If E is the event of getting a red and a black ball in a draw, the number of cases favourable to $E = 4 \times 5 = 20$

$\therefore$ The required probability is $\frac{20}{105} = \frac{4}{21}$

4. Example : Ten dice are thrown. Find the probability that none of the dice shows the number 1.

Solution : We can express the sample space of this experiment as a list of 10 - tuples formed with the symbols 1, 2, 3, 4, 5 and 6. (An n-tuple is an orderly arrangement of n numbers). There are 6^{10} such entries, all of which are equally likely.

Let A be the event that none of the dice shows the number 1. The number of outcomes that do not have the number 1 is the number of 10 - tuples whose elements are chosen from the symbols 2, 3, 4, 5 and 6. The number of such 10 - tuples is 5^{10}. Therefore

$$P(A) = \frac{5^{10}}{6^{10}} = \left(\frac{5}{6}\right)^{10}$$

9.2.3 Limitations of the classical definition of probability

The classical definition of probability has the following limitations :

(i) If the outcomes of a random experiment are not equally likely, then the probability of an event in such an experiment is not defined.

For instance, if a bag contains five red and seven black balls then the event of picking a red ball and picking a black ball from the bag are not equally likely.

(ii) If the random experiment contains infinitely many outcomes, then this definition can not be applied to find the probability of an event in such an experiment. For example, either in the random experiment of tossing a coin until tail appears or choosing a natural number, there are infinite number of outcomes.

In order to overcome these deficiencies, we now consider the relative frequency approach to the definition of probability.

9.2.4 Relative frequency (or statistical or emperical) definition of probability

Suppose a random experiment is repeated n times, out of which an event E occurs m(n) times. Then the ratio $r_n = \dfrac{m(n)}{n}$ *is called* ***the n-th relative frequency*** *of the event E.*

Now consider the sequence $r_1, r_2, ..., r_n, ...$ *If* r_n *tends to a definite limit, say l, as n tends to infinity*

i.e., $\lim\limits_{n\to\infty} r_n = l$, *then l is defined to be the probability of the event E and we write*

$$P(E) = \lim_{n\to\infty} r_n = l$$

9.2.5 Deficiencies of the relative frequency definition of probability

From the above definition we observe the following deficiencies:

(i) Repeating a random experiment infinitely many times is practically impossible.

(ii) The sequence of relative frequencies is assumed to tend to a definite limit, which may not exist.

(iii) The values $r_1, r_2, ..., r_n$ are not real variables. It is therefore not possible to prove the existence and the uniqueness of the limit of r_n as $n \to \infty$, by applying methods used in calculus.

9.2.6 Solved Problems

1. Problem : *A number x is drawn arbitrarily from the set* $\{1,2,3,......,100\}$.

Find the probability that $\left(x+\frac{100}{x}\right) > 29$

Solution : The total points of the sample space are 100. Let A be the event that an x selected (drawn) at random from the set $S = \{1,2,3,......,100\}$ has the property $\left(x+\frac{100}{x}\right) > 29$

Now $x+\frac{100}{x} > 29 \Leftrightarrow x^2 - 29x + 100 > 0 \Leftrightarrow (x-4)\,(x-25) > 0 \Leftrightarrow x > 25 \quad or \quad x < 4$

Since $x \in S$, it follows that $A = \{1, 2, 3, 26, 27, ..., 100\}$

Thus the number of cases favourable to A is 78

$\therefore$ The required probability : $P(A) = \frac{78}{100} = 0.78$.

2. Problem : *Two squares are chosen at random on a chess board. Show that the probability that they have a side in common is* $\frac{1}{18}$.

Solution : The number of ways of choosing the first square is 64 and that of the second is 63. Therefore the number of ways of choosing the first and the second square is 64×63.

Let E be the event that these squares have a side in common. We shall find the number of cases favourable to E.

If the first square happens to be one of the squares in the four corners of the chessboard, the second square (with common side) can be chosen in 2 ways.

If the first square happens to be any one of the remaining 24 squares along the four sides of the chess board other than the corner, the second square can be chosen in 3 ways.

If the first square happens to be any one of the remaining 36 inner squares, then the second square can be chosen in 4 ways.

Hence the number of cases favourable to E is $(4 \times 2) + (24 \times 3) + (36 \times 4) = 224$

Therefore the required probability is $\frac{224}{64 \times 63} = \frac{1}{18}$.

3. Problem : *A fair coin is tossed 200 times. Find the probability of getting a head an odd number of times.*

Solution : The total number of points in the sample space is 2^{200}. Let E be event of getting a head an odd number of times. Then the number of cases favourable to E is

$$^{200}C_1 + {}^{200}C_3 + \cdots\cdots + {}^{200}C_{199} = \frac{2^{200}}{2} = 2^{199}$$

$$\therefore \text{ The required probability } P(E) = \frac{2^{199}}{2^{200}} = \frac{1}{2}.$$

4. Problem : *A and B are among 20 persons who sit at random along a round table. Find the probability that there are any six persons between A and B.*

Solution : Let A occupy any seat at the round table. Then there are 19 seats left for B. But if six persons are to be seated between A and B, then B has only two ways to sit. Thus the required probability is $\frac{2}{19}$.

5. Problem : *Out of 30 consecutive integers, two integers are drawn at random. Find the probability that their sum is odd.*

Solution : The total number of ways of choosing 2 integers out of 30 is $^{30}C_2$. Out of the 30 numbers, 15 are even and 15 are odd. If the sum of the two numbers is to be odd, one should be even and the other odd. Hence the number of cases favourable to the required event is $^{15}C_1 \times {}^{15}C_1$.

$$\therefore \text{ The required probability } = \frac{^{15}C_1 \times {}^{15}C_1}{^{30}C_2} = \frac{15 \times 15 \times 2}{30 \times 29} = \frac{15}{29}.$$

6. Problem : *Out of 1,00,000 new born babies, 77,181 survived till the age of 20. Find the probability that a new born baby survives till 20 years of age.*

Solution

Here $m = 77{,}181$; $n = 1{,}00{,}000$

$$\text{Required probability } = \frac{77{,}181}{1{,}00{,}000} = 0.77181.$$

9.2.7 Axiomatic approach to probability

Let S be the sample space of a random experiment $\mathscr{E}$. Then the set of all subsets of S is called **the power set of S** and is denoted by $\mathscr{P}(S)$. Observe that the set of all possible events of this experiment is the power set $\mathscr{P}(S)$.

We now introduce Kolmogorov's axiomatic approach to the theory of probability, which overcomes the short comings (deficiencies) of both the definitions 9.2.1 and 9.2.4.

9.2.8 Definition

Let S *be the sample space of a random experiment* $\mathscr{E}$ *which is finite . Then a function* $P : \mathscr{P}(S) \to \mathbf{R}$ *satisfying the following axioms is called a probability function* :

(i) $P(E) \geq 0 \ \forall E \in \mathscr{P}(S)$ (axiom of non-negativity)

(ii) $P(S) = 1$ (axiom of certainty)

(iii) If $E_1, E_2 \in \mathscr{P}(S)$ and $E_1 \cap E_2 = \phi$ then,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) \quad \text{(axiom of additivity)}$$

For each $E \in \mathscr{P}(S)$, the real number P(E) is called the probability of the event E. If $E = \{a\}$, then we write $P(a)$ instead of $P(\{a\})$.

1. Note : $P(\phi) = 0$ for any sample space S, $S \cup \phi = S$, $S \cap \phi = \phi$,

$$P(S) = P(S \cup \phi) = P(S) + P(\phi) \Rightarrow P(\phi) = 0$$

2. Note : If S is countably infinite, then axiom (iii) of the above definition is to be replaced by (iii)* : If $\{E_n\}_{n=1}^{\infty}$ is a sequence of pairwise mutually exclusive events, then

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n) \qquad \text{(axiom of countable additivity)}$$

3. Note : Suppose S is the sample space of a random experiment $\mathscr{E}$. Let P be a probability function. If $E_1, E_2, ..., E_n$ are finitely many pairwise mutually exclusive events, then

$$P(E_1 \cup E_2 \cup ..., \cup E_n) = P(E_1) + P(E_2) + \cdots + P(E_n)$$

This can be proved by Mathematical Induction. In particular if E has n-elements,

$$P(E) = \sum_{a \in E} P(a).$$

9.2.9 Examples

1. Example

Let $S = \{1, 2, 3, 4\}$ be the sample space of a random experiment $\mathscr{E}$.

Suppose $P(1) = 0.1$, $P(2) = 0.2$, $P(3) = 0.3$ and $P(4) = 0.4$

Define $P(A) = \sum_{a \in A} P(a)$ for any subset A of S. Then P is a probability function.

We observe here that

(i) $P(A) > 0$, if $\phi \neq A \subset S$

(ii) $P(S) = \sum_{a \in S} P(a) = P(1) + P(2) + P(3) + P(4)$

$= 0.1 + 0.2 + 0.3 + 0.4 = 1$

(iii) If $A, B \subset S$ and $A \cap B = \phi$

then $P(A \cup B) = \sum_{a \in A \cup B} P(a) = \sum_{a \in A} P(a) + \sum_{a \in B} P(a) \quad (\because A \cap B = \phi)$

$= P(A) + P(B)$

Hence P is a probability function.

2. Example : Suppose $S = \{0, 1, 2, 3, 4\}$ is the sample space of a random experiment $\mathscr{E}$. Define

$P(0) = 0,\ P(1) = 0,\ P(2) = \frac{1}{2},\ P(3) = \frac{1}{4}$, and $P(4) = \frac{1}{4}$. Define $P(A) = \sum_{a \in A} P(a)$ for $A \subseteq S$

Then P defines a probability function. Here

(i) $P(A) \geq 0 \quad \forall a \in S$.

(ii) $\sum_{a \in S} P(a) = P(0) + P(1) + P(2) + P(3) + P(4) = 0 + 0 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$

If $A \subseteq S$ and $B \subseteq S$, $A \cap B = \phi$ then

$P(A \cup B) = \sum_{a \in A \cup B} P(a) = \sum_{a \in A} P(a) + \sum_{a \in B} P(a) = P(A) + P(B)$

9.2.10 Theorem

Let S be sample space of a random experiment $\mathscr{E}$ and P be a probability function on $\mathscr{P}(S)$. Then

(i) $P(\phi) = 0$

(ii) *If E^c is the complementary event of E, then* $P(E^c) = 1 - P(E)$

(iii) $0 \leq P(E) \leq 1,\ \forall E \subseteq S$.

(iv) *If* $E_1 \subseteq E_2$, *then* $P(E_2 - E_1) = P(E_2) - P(E_1)$

(v) *If* $E_1 \subseteq E_2$, *then* $P(E_1) \leq P(E_2)$

Proof : (i) We know that $S \cap \phi = \phi$ and $S \cup \phi = S$

Hence $P(S) = P(S \cup \phi) = P(S) + P(\phi)$

$1 = 1 + P(\phi)$, so that $P(\phi) = 0$

(ii) We have $E \cap E^c = \phi$ and $E \cup E^c = S$

Hence $P(S) = P(E \cup E^c) = P(E) + P(E^c)$

$$1 = P(E) + P(E^c), \text{ so that } P(E^c) = 1 - P(E)$$

(iii) By the definition of P, we have $P(E) \geq 0$ and $P(E^c) \geq 0$

Then by (ii), $P(E) \leq 1$

$\therefore 0 \leq P(E) \leq 1$

(iv) If $E_1 \subseteq E_2$, then we have $E_2 = (E_2 - E_1) \cup E_1$ and $(E_2 - E_1) \cap E_1 = \phi$

Hence $P(E_2) = P\big((E_2 - E_1) \cup E_1\big) = P(E_2 - E_1) + P(E_1)$

$\therefore P(E_2 - E_1) = P(E_2) - P(E_1)$

(v) Since $E_1 \subseteq E_2$, we have $P(E_2) - P(E_1) = P(E_2 - E_1) \geq 0$

$$\therefore P(E_1) \leq P(E_2)$$

9.2.11 Theorem (Addition theorem on probability)

Statement : If E_1, E_2 *are any two events of a random experiment and* P *is a probability function, then*

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Proof

Case (i) : Suppose $E_1 \cap E_2 = \phi$ (see Fig. 9.1)

Then $P(E_1 \cap E_2) = 0$ and the result follows from axiom (iii) of definition (9.2.8)

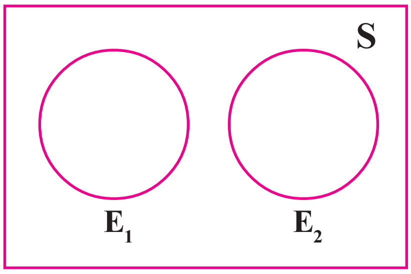


Fig. 9.1

Case (ii) : Suppose $E_1 \cap E_2 \neq \phi$. Then

$E_1 \cup E_2 = E_1 \cup (E_2 - E_1)$ and

$E_1 \cap (E_2 - E_1) = \phi$ (see Fig. 9.2)

$\therefore P(E_1 \cup E_2) = P\big(E_1 \cup (E_2 - E_1)\big)$

$= P(E_1) + P(E_2 - E_1)$

$= P(E_1) + P(E_2 - (E_1 \cap E_2))$

since $E_2 - E_1 = (E_2 - (E_1 \cap E_2))$,

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$: by 9.2.10 (iv)

$\therefore P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

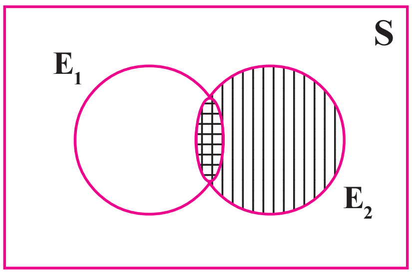


Fig. 9.2

$E_1 \cap E_2$

$E_2 - E_1$

9.2.12 Remarks

(i) In the discussion that follows, unless otherwise specified, by a probability function we mean a function that satisfies all the properties mentioned in the definition (9.2.8)

(ii) Suppose that the sample space of a random experiment is a finite set S and $p : S \to \mathbf{R}$ be such that $p(a) \geq 0$ for all $a \in S$ and $\sum_{a \in S} p(a) = 1$. For any event $E(\subset S)$, define

$$P(E) = \begin{cases} \sum_{a \in S} p(a), & \text{if } E \neq \phi \\ 0, & \text{if } E = \phi \end{cases}$$

Thus any function p defined from S into the set of non-negative real numbers satisfying $\sum_{a \in S} p(a) = 1$ defines a probability function.

(iii) Some set - theoretic descriptions of various events, useful in solving problems on probability are listed below :

No.	Event	Set - theoretic description
1.	Event A or Event B to occur	$A \cup B$
2.	Both event A and B to occur	$A \cap B$
3.	Neither A nor B occurs	$(A \cup B)^c = A^c \cap B^c$
4.	A occurs but B does not occur	$A \cap B^c$ or $A \backslash B$
5.	Exactly one of the events A, B to occur	$(A \cap B)^c \cup (A^c \cap B)$ or $(A\text{-}B) \cup (B\text{-}A)$ or $(A \cup B) - (A \cap B)$
6.	Not more than one of the events A, B occurs	$(A \cap B)^c \cup (A^c \cap B) \cup (A^c \cap B^c)$
7.	Event B occurs whenever A occurs	$A \subseteq B$

9.2.13 Solved Problems

1. Problem : *Find the probability of throwing a total score of 7 with 2 dice.*

Solution : The sample space S of this experiment is given by

$$S = \{ (1,1), (1,2), \ldots, (1,6),$$
$$(2,1), (2,2), \ldots, (2,6),$$
$$\vdots$$
$$(6,1), (6,2), \ldots, (6,6) \}$$

In a typical element the first coordinate represents the score on the first die and the second coordinate represents the score on the second die. There are 36 points in S and all these points are equally likely. Let E be the event of getting a total score of 7. Then E has the following 6 elements.

$$\{ (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) \}$$

$$\therefore \; P(E) = \frac{6}{36} = \frac{1}{6}$$

2. Problem : *Find the probability of obtaining two tails and one head when 3 coins are tossed.*

Solution : For this experiment of tossing three coins, the sample space can be seen to be

$$S = \{ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT \}$$

Let E be the event of obtaining two tails and a head.

Then $E = \{HTT, THT, TTH\}$ $\quad\therefore\; P(E) = \frac{3}{8}$

3. Problem : *A page is opened at random from a book containing 200 pages. What is the probability that the number on the page is a perfect square.*

Solution : The sample space S of the experiment in question is given by $S = \{ 1, 2, 3, ..., 200)$, so that the number of points in the sample space $n(S) = 200$. Let E be the event of drawing a page whose number is a perfect square. Then $E = \{ 1, 4, 9, ..., 196\}$ so that $n(E) = 14$

$$\therefore \; P(E) = \frac{n(E)}{n(S)} = \frac{14}{200} = \frac{7}{100}$$

4. Problem : *Find the Probability of drawing an Ace or a spade from a well shuffled pack of 52 playing cards.*

A pack of cards contains a total of 52 cards. These 52 cards are divided into 4 sets namely Hearts, Diamonds, Spades and Clubs. Each set consists of 13 cards, namely :

A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

(Here A : Ace, K : King, Q : Queen, J : Jack)

Solution : Let E_1 be the event of drawing a spade and E_2 be the event of drawing an Ace. Observe here that E_1, E_2 are not mutually exclusive.

Now $n(E_1) = 13$, $n(E_2) = 4$ and $n(E_1 \cap E_2) = 1$

$$\therefore P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

5. Problem : *If A and B are two events then show that* (i) $P(A \cap B^c) = P(A) - P(A \cap B)$ *and* (ii) *the probability that one of them occurs is given by* $P(A) + P(B) - 2\ P(A \cap B)$

Solution

(i) We have $A = (A \cap B) \cup (A \cap B^c)$ and $(A \cap B) \cap (A \cap B^c) = \phi$

Hence $P(A) = P(A \cap B) + P(A \cap B^c)$

$\therefore\ P(A \cap B^c) = P(A) - P(A \cap B)$

(ii) Let E be the event that exactly one of them, i.e., either A or B occurs. Then,

$$E = (A - B) \cup (B - A)$$
$$= (A \cap B^c) \cup (B \cap A^c)$$

So $P(E) = P(A \cap B^c) + P(B \cap A^c)$, (Since $(A \cap B^c) \cap (B \cap A^c) = \phi$

$$= P(A) - P(A \cap B) + P(B) - P(B \cap A), \quad \text{(using (i))}$$
$$= P(A) + P(B) - 2P(A \cap B)$$

6. Problem : *A and B are events with* $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cap B) = 0.3$. *Find the probability that (i) A does not occur (ii) neither A nor B occurs.*

Solution

(i) We know that A^c denotes the event: A does not occur and $(A \cup B)^c$ denotes the event : neither A nor B occurs. Then

$$P(A^c) = 1 - P(A) = 1 - 0.5 = 0.5$$

(ii) By addition theorem $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.5 + 0.4 - 0.3 = 0.6$$

$$\therefore\ P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.6 = 0.4$$

7. Problem : *If A, B, C are three events show that*

$$\therefore\ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Solution : Write $B \cup C = D$, Then

$$P(A \cup B \cup C) = P(A \cup D) = P(A) + P(D) - P(A \cap D)$$
$$= P(A) + P(B \cup C) - P\big((A \cap (B \cup C)\big)$$
$$= P(A) + P(B) + P(C) - P(B \cap C) - P[(A \cap B) \cup (A \cap C)] \qquad \ldots (1)$$

Let $E = (A \cap B)$ and $F = (A \cap C)$. Then

$$P((A \cap B) \cup (A \cup C)) = P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap A \cap C)$$

$$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \qquad \ldots (2)$$

Using (2) in (1), we get the desired result.

Exercise 9(b)

I. 1. If 4 fair coins are tossed simultaneously, then find the probability that 2 heads and 2 tails appear.

2. Find the probability that a non-leap year contains (i) 53 Sundays (ii) 52 Sundays only.

3. Two dice are rolled. What is the probability that none of the dice shows the number 2?

4. In an experiment of drawing a card at random from a pack, the event of getting a spade is denoted by A and getting a pictured card (King, Queen or Jack) is denoted by B. Find the probabilities of $A, B, A \cap B$ and $A \cup B$.

5. In a class of 60 boys and 20 girls, half of the boys and half of the girls know cricket. Find the probability of the event that a person selected from the class is either a boy, or a girl knowing cricket.

6. For any two events A and B, show that $P(A^c \cap B^c) = 1 + P(A \cap B) - P(A) - P(B)$.

7. Two persons A and B are rolling a die on the condition that the person who gets 3 will win the game. If A starts the game, then find the probabilities of A and B respectively to win the game.

8. A,B,C are 3 news papers from a city. 20% of the population read A, 16% read B, 14% read C, 8% both A and B, 5% both A and C, 4% both B and C and 2% all the three. Find the percentage of the population who read atleast one news paper.

9. If one ticket is randomly selected from tickets numbered 1 to 30, then find the probability that the number on the ticket is (i) a multiple of 5 or 7 (ii) a multiple of 3 or 5.

10. If two numbers are selected randomly from 20 consecutive natural numbers, find the probability that the sum of the two numbers is (i) an even number (ii) an odd number.

11. A game consists of tossing a coin 3 times and noting its outcome. A boy wins if all tosses give the same outcome and loses otherwise. Find the probability that the boy loses the game.

12. If E_1, E_2 are two events with $E_1 \cap E_2 = \phi$, then show that $P(E_1^C \cap E_2^C) = P(E_1^C) - P(E_2)$.

II. 1. A pair of dice is rolled 24 times. A person wins by not getting a pair of 6s on any of the 24 rolls. What is the probability of his winning? (This is the problem posed by the French gambler Chevalier de Mere (1607-1685) to Blaise Pascal, who inturn discussed it with Pierre de Fermat and solved the same).

2. If P is probability function, then show that for any two events A and B,
$P(A \cap B) \le P(A) \le P(A \cup B) \le P(A) + P(B)$

3. In a box containing 15 bulbs, 5 are defective . If 5 bulbs are selected at random from the box, find the probability of the event, that
   (i) none of them is defective
   (ii) only one of them is defective
   (iii) atleast one of them is defective.

4. A and B are seeking admission into IIT. If the probability for A to be selected is 0.5 and that both to be selected is 0.3, then is it possible that, the probability of B to be selected is 0.9 ?

5. The probability for a contractor to get a road contract is $\frac{2}{3}$ and to get a building contract is $\frac{5}{9}$. The probability to get atleast one contract is $\frac{4}{5}$. Find the probability that he gets both the contracts.

6. In a committee of 25 members, each member is proficient either in mathematics or in statistics or in both. If 19 of these are proficient in mathematics, 16 in statistics, find the probability that a person selected from the committee is proficient in both.

7. A,B,C are three horses in a race. The probability of A to win the race is twice that of B, and probability of B is twice that of C. What are the probabilities of A, B, and C to win the race?

8. A bag contains 12 two rupee coins, 7 one rupee coins and 4 half a rupee coins. If three coins are selected at random, then find the probability that
   (i) the sum of three coins is maximum
   (ii) the sum of three coins is minimum
   (iii) each coin is of different value

9. The probabilities of three events A,B,C are such that

$P(A)=0.3,\ P(B)=0.4, P(C)=0.8,\quad P(A\cap B)=0.08, P(A\cap C)=0.28, P(A\cap B\cap C)=0.09$

and $P(A\cup B\cup C)\geq 0.75$

Show that $P(B\cap C)$ lies in the interval $[0.23, 0.48]$

10. The probabilities of three mutually exclusive events are respectively given as $\frac{1+3p}{3}, \frac{1-p}{4}, \frac{1-2p}{2}$. Prove that $\frac{1}{3}\leq p\leq\frac{1}{2}$

11. On a festival day, a man plans to visit 4 holy temples A, B, C, D in a random order. Find the probability that he visits (i) A before B (ii) A before B and B before C.

12. From the employees of a company, 5 persons are selected to represent them in the managing committee of the company. The particulars of 5 persons are as follows :

S.No.	Name	Sex	Age in years
1.	Harish	M	30
2	Rohan	M	33
3.	Sheetala	F	46
4.	Alis	F	28
5.	Salim	M	41

A person is selected at random from this group to act as a spokesperson. Find the probability that the spokesperson will be either male or above 35 years.

13. Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, find the probability that (i) you both enter the same section (ii) you both enter the different sections.

9.3 Independent and Dependent Events, Conditional Probability, Multiplication Theorem and Baye's Theorem

Suppose a coin is tossed three times and we are interested in the probability of the event of appearing an odd number of tails of which the first one is a tail. Let F be the event of getting a tail in the first toss. Let E be the event that an odd number of tails appear. Since the first flip comes up tails, there are only four out of eight possible outcomes for our event : TTT, TTH, THT, THH. Also, an odd number of tails appears only for the outcomes TTT and THH. Since the eight outcomes have equal probability, each of the above four outcomes, given that F occurs, should also have an equal probability of $\frac{1}{4}$. This suggests that we should assign the probability $\frac{2}{4}=\frac{1}{2}$ to the event "E, given that F has occurred". This probability is called the conditional probability of E given F and we denote it by P(E|F).

We now ask the question : Does, knowing that the first flip comes up tails (event F) alter or influence the probability that tails come up an odd number of times (event E)? In other words, is it the case that P(E|F) = P(E)? This equality is valid for the above events E and F, since $P(E|F)=\frac{1}{2}$ and $P(E)=\frac{1}{2}$. Hence E and F are independent events.

In this section we shall first define a conditional event, conditional probability and establish the multiplication theorem of probability. We then define independent and dependent events and finally state and prove the Baye's theorem.

9.3.1 Definition (Conditional Event)

Suppose A and B are two possible events of a random experiment. If the event B occurs after the occurrence of the event A, then the event : "happening of B after the happening of A" is called a conditional event and is denoted by B|A. Similarly A|B stands for the event : happening of A after the happening of B.

9.3.2 Definition (Conditional Probability)

If A and B are two events of a sample space S and $P(A)\neq 0$, then the probability of B after the event A has occurred is called the conditional probability of B given A, and is denoted by P (B|A) : We define

$$P(B|A)=\frac{P(B\cap A)}{P(A)};\quad P(A)\neq 0$$

9.3.3 Examples

1. Example : *A pair of dice is thrown. Find the probability that either of the dice shows 2 when their sum is 6.*

As has already been shown in 9.2.13, problem 1, the sample space of this experiment consists of 36 ordered pairs (a, b) where a and b can be any of the integers 1 to 6. We have to find P(A|B), where

A : 2 appears on either of the dice and

B : the sum of the two numbers on the dice is 6.

Since $A = \{(2,1), (2,2), (2,3) .... (2,6), (1,2), (2,2), ...(6,2)\}$,

$B = \{(1,5), (2,4), (3,3), (4,2),(5,1)\}$, we have $P(B) = \frac{5}{36}$

Out of the five elements of B, (2,4), (4,2) lie in A,

i.e., $A \cap B = \{(2,4),(4,2)\}$, so that $P(A \cap B) = \frac{2}{36}$

$$\therefore \; P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{5/36} = 2/5$$

2. Example : A box contains 4 defective and 6 good bulbs. Two bulbs are drawn at random without replacement. Let us find the probability that both bulbs drawn are good. Let A denote the event of drawing a good bulb in the first draw and B denote the event that the second bulb drawn is also good.

Then, $P(A) = \frac{6}{10}$ and $P(B|A) = \frac{5}{9}$

P (both the bulbs drawn are good) $= P(A \cap B) = P(A)\ P(B|A) = \frac{6}{10} . \frac{5}{9} = \frac{1}{3}$

9.3.4 Theorem (Multiplication theorem of probability)

If A and B are two events of a random experiment with $P(A) > 0$ *and* $P(B) > 0$,

then $P(A \cap B) = P(A)\ P(B|A) = P(B)\ P(A|B)$

Proof : Let S be the sample space associated with the random experiment. Let A, B be two events of S such that $P(A) > 0$ and $P(B) > 0$. Then by the definition of conditional probability,

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$\therefore \; P(B \cap A) = P(A).P(B|A)$$

since $P(B) > 0$ we interchange A, B in the above and get

$$P(A \cap B) = P(B \cap A) = P(B).P(A|B)$$

The following is an extension of the above theorem which can be proved by induction.

9.3.5 Theorem

If $A_1, A_2, ..., A_m$ *are m events of a random experiment with* $P\left(\bigcap_{i=1}^{m-1} A_i\right) > 0$, *then*

$$P(A_1 \cap A_2 \cap ... \cap A_m) = P(A_1)\, P(A_2|A_1)\, P(A_3|A_1 \cap A_2) ... P(A_m|A_1 \cap A_2 \cap ... \cap A_{m-1}).$$

9.3.6 Examples

1. Example : A bag contains 10 identical balls, of which 4 are blue and 6 are red. Three balls are taken out at random from the bag one after the other. Let us find the probability that all the three balls drawn are red.

Since six out of the ten balls are red, the probability that the first ball drawn is red is $\frac{6}{10}$. If the first ball drawn is red, 9 balls are left and 5 out of them are red. Similarly if the first two balls drawn are red coloured, then the probability that the third ball drawn is red is $\frac{4}{8}$.

Hence by the multiplication theorem, the required probability is $\frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} = \frac{1}{6}$

2. Example : An urn contains 7 red and 3 black balls. Two balls are drawn without replacement. What is the probability that the second ball is red if it is known that the first ball drawn is red.

Let R_1 be the event of drawing the first ball red and R_2 be the event of drawing the second ball also red. Then $P(R_1) = \frac{7}{10}$. But, after one red ball is drawn out, 6 red balls and 3 black balls remain in the urn. Therefore the probability $P(R_2|R_1) = \frac{6}{9} = \frac{2}{3}$

Note : Suppose S is the sample space of a random experiment. Let P be a probability function on $\mathscr{P}(S)$, the set of all events of the random experiment. Let $A \in \mathscr{P}(S)$ and $P(A) > 0$. Then $P_o : \mathscr{P}(S) \to \mathbf{R}$ defined by $P_0(E) = P(E \mid A)$ $E \in \mathscr{P}(S)$ is also a probability function on $\mathscr{P}(S)$.

9.3.7 Definition

Let S be the sample space of a random experiment and P *be a probability function defined on P(S). Then two events A and B of the experiment are called independent (or stochastically independent) if* $P(A \cap B) = P(A).P(B)$.

If A and B are independent, then $P(A/B) = P(A)$ if $P(B) \neq 0$. That is, the conditional probability of A, given B, is the same as the "absolute" probability of A. The fact that B has occured does not have any influence on the probability of occurence of A. Similarly, $P(B/A) = P(B)$ if $P(A) \neq 0$ and A, B are independent events.

Events A, B and C of a random experiment are called independent if they are pairwise independent and $P(A \cap B \cap C) = P(A).P(B).P(C)$.

9.3.8 Examples

1. Example : A bag contains 10 balls; 5 of which are red and the remaining blue. Two balls are drawn at random from the bag one after the other with replacement. Let A be the event that the first ball drawn is red and B be the event that the second ball is red. We shall find whether these events are independent.

Since the first ball drawn is replaced before drawing the second ball, there are $10 \times 10 = 100$ ways of picking the two balls. Out of these 100, $5 \times 10 = 50$ draws have the property that the first ball is red. Thus $P(A) = \frac{50}{100} = \frac{1}{2}$. Similarly $P(B) = \frac{1}{2}$. Also, there are $5 \times 5 = 25$ ways of drawing two balls such that the first and second are both red balls.

$$\therefore \ P(A \cap B) = \frac{25}{100} = \frac{1}{4}$$

But $P(A \cap B) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(A).P(B)$.

Hence events A and B are independent. This outcome makes sense because the colour of the ball we observe on the second draw does not in any way depend on the colour of the first ball.

2. Example : Let A and B be independent events with $P(A) = 0.2$, $P(B) = 0.5$. Let us find

(i) $P(A|B)$ (ii) $P(B|A)$ (iii) $P(A \cap B)$ and (iv) $P(A \cup B)$

Since A, B are independent, we have

(i) $P(A|B) = P(A) = 0.2$ and

(ii) $P(B|A) = P(B) = 0.5$

(iii) $P(A \cap B) = P(A)\,P(B) = 0.2 \times 0.5 = 0.1$

(iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.2 + 0.5 - 0.1 = 0.6$

3. Example : Bag B_1 contains 4 white and 2 black balls. Bag B_2 contains 3 white and 4 black balls. A bag is drawn at random and a ball is chosen at random from it. What is the probability that the ball drawn is white?

Let E_1, E_2 denote the events of choosing bags B_1 and B_2 respectively.

$$\text{Then } P(E_1) = P(E_2) = \frac{1}{2}$$

Let W be the event that the ball chosen from the selected bag is white.

$$\text{Then } P(W|E_1) = \frac{4}{6} = \frac{2}{3}. \qquad \text{Similarly } P(W|E_2) = \frac{3}{7}$$

Observe that $W = (W \cap E_1) \cup (W \cap E_2)$ and $(W \cap E_1) \cap (W \cap E_2) = \varnothing$

$$\therefore P(W) = P(W \cap E_1) + P(W \cap E_2)$$

$$= P(E_1).P(W|E_1) + P(E_2).P(W|E_2) = \frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{3}{7} = \frac{1}{3} + \frac{3}{14} = \frac{23}{42}$$

9.3.9 Theorem (Baye's Theorem)

Suppose $E_1, E_2, ..., E_n$ *are n mutually exclusive and exhaustive events of a random experiment with* $P(E_i) \neq 0$, *for i = 1, 2, ..., n. Then for any event* A *of the random experiment with* $P(A) \neq 0$,

$$P(E_k|A) = \frac{P(E_k)P(A|E_k)}{\sum\limits_{i=1}^{n} P(E_i)P(A|E_i)}, \qquad k = 1, 2, ..., n$$

Proof : Given that $P(E_i) > 0$ for $i = 1, 2, ...n$. By hypothesis. $i \neq j$, $E_i \cap E_j = \phi$ and $\bigcup\limits_{i=1}^{n} E_i = S$, the sample space of the experiment. Since $A \subseteq S$ for any event A, we have

$$A = A \cap S = A \cap \left(\bigcup_{i=1}^{n} E_i\right) = \bigcup_{i=1}^{n} (A \cap E_i)$$

Also, for $i \neq j, (A \cap E_i) \cap (A \cap E_j) = A \cap (E_i \cap E_j) = A \cap \phi = \phi$

Therefore $P(A) = \sum\limits_{i=1}^{n} P(A \cap E_i) = \sum\limits_{i=1}^{n} P(E_i).P(A|E_i)$, (by using multiplication theorem)

$$\text{Hence} \quad P(E_k|A) = \frac{P(E_k \cap A)}{P(A)} = \frac{P(E_k)P(A|E_k)}{\sum\limits_{i=1}^{n} P(E_i)P(A|E_i)}$$

9.3.10 Examples

1. Example : A shop keeper buys a particular type of electric bulbs from three manufacturers M_1, M_2 and M_3. He buys 25% of his requirement from M_1, 45% from M_2 and 30% from M_3. Based on the past experience, he found that 2% of type M_3 bulbs are defective, where as only 1% of type M_1 and Type M_2 are defective. If a bulb chosen by him at random is found defective, let us find the probability that it was of type M_3.

If E is the event that the bulb chosen is defective, then $P(M_3|E)$ is the required probability. By hypothesis, we have $P(M_1) = 0.25$, $P(M_2) = 0.45$ and $P(M_3) = 0.3$

Then $P(E|M_1) = 0.01$, $P(E|M_2) = 0.01$ and $P(E|M_3) = 0.02$

We have by Baye's theorem,

$$P(M_3|E) = \frac{P(E|M_3).\,P(M_3)}{P(E|M_1)P(M_1)+P(E|M_2)P(M_2)+P(E|M_3)P(M_3)}$$

$$= \frac{0.02\times0.3}{(0.01\times0.25)+(0.01\times0.45)+(0.2\times0.3)} = 0.46$$

2. Example : Suppose that an urn B_1 contains 2 white and 3 black balls and another urn B_2 contains 3 white and 4 black balls. One urn is selected at random and a ball is drawn from it. If the ball drawn is found black, let us find the probability that the urn chosen was B_1.

Let E_1, E_2 denote the events of selecting urns B_1 and B_2 respectively. Then $P(E_1) = P(E_2) = \frac{1}{2}$.

Let B denote the event that the ball chosen from the selected urn is black. Then we have to find $P(E_1|B)$.

By hypothesis $P(B|E_1) = \frac{3}{5}$; $P(B|E_2) = \frac{4}{7}$.

By Baye's theorem, $P(E_1|B) = \dfrac{P(E_1)P(B|E_1)}{P(E_1).P(B|E_1)+P(E_2)P(B|E_2)}$

$$= \frac{\frac{1}{2}\times\frac{3}{5}}{\left(\frac{1}{2}\times\frac{3}{5}\right)+\left(\frac{1}{2}\times\frac{4}{7}\right)} = \frac{21}{41}.$$

9.3.11 Solved Problems

1. Problem : *Suppose there are 12 boys and 4 girls in a class. If we choose three children one after another in succession at random, find the probability that all the three are boys.*

Solution : Let E_i be the event of choosing a male child in i^{-th} trial, $i = 1, 2, 3$. We have to find $P(E_1 \cap E_2 \cap E_3)$.

$$\text{Here } P(E_1) = \frac{3}{4}, \quad P(E_2|E_1) = \frac{11}{15}, \quad P(E_3|E_2 \cap E_1) = \frac{5}{7}$$

By the multiplication theorem,

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1 \cap E_2) = \frac{3}{4} \times \frac{11}{15} \times \frac{5}{7} = \frac{11}{28}$$

2. Problem : *A speaks truth in 75% of the cases and B in 80% cases. What is the probability that their statements about an incident do not match.*

Solution : Let E_1, E_2 be the events that A and B respectively speak truth about an incident. Then

$$P(E_1) = \frac{75}{100} = \frac{3}{4}, \quad P(E_2) = \frac{80}{100} = \frac{4}{5}, \quad \text{so that } P(E_1^c) = \frac{1}{4}, \; P(E_2^c) = \frac{1}{5}.$$

If E be the event that their statements do not match about the incident, then this happens in two mutually exclusive ways :

(i) A speaks truth and B tells lie (ii) A tells lie and B speaks truth

These two events can be represented respectively by $E_1 \cap E_2^c$ and $E_1^c \cap E_2$

$$\therefore \; P(E) = P(E_1 \cap E_2^c) + P(E_1^c \cap E_2)$$

$$= P(E_1).P(E_2^c) + P(E_1^c)\, P(E_2), \; \text{(Since } E_1, E_2 \text{ are independent events)}$$

$$= \left(\frac{3}{4} \times \frac{1}{5}\right) + \left(\frac{1}{4} \times \frac{4}{5}\right) = \frac{7}{20}.$$

3. Problem : *A problem in Calculus is given to two students A and B whose chances of solving it are $\frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability of the problem being solved if both of them try independently.*

Solution : Let E_1 and E_2 denote the events that the problem is solved by A and B respectively. We are given that $P(E_1) = \frac{1}{3}$ and $P(E_2) = \frac{1}{4}$. We have to find $P(E_1 \cup E_2)$.

By Addition Theorem. $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$$= P(E_1) + P(E_2) - P(E_1)P(E_2))$$

$(\therefore\ E_1, E_2$ are independent)

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{1}{2}$$

4. Problem : *A and B toss a coin 50 times each simultaneously. Find the probability that both of them will not get tails at the same toss.*

Solution : Let E be the event that A and B both will not get tails at the same toss. In each toss we have the following four choices:

(i) A : Head, B : Head (ii) A : Head, B : Tail

(iii) A : Tail, B : Head (iv) A : Tail, B : Tail

Since there are 50 trails, the total number of choices is 4^{50}. But out of the four choices listed above, only (i), (ii) and (iii) are favourable to the occurrence of the required event E.

$$\therefore\ P(E) = \frac{3^{50}}{4^{50}} = \left(\frac{3}{4}\right)^{50}$$

5. Problem : *If A and B are independent events of a random experiment, show that A^c and B^c are also independent.*

Solution : If A and B are independent, then $P(A \cap B) = P(A)P(B)$

Now $P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$

$$= 1 - [P(A) + P(B) - P(A).P(B)] = (1 - P(A))(1 - P(B)) = P(A^c).P(B^c)$$

$\therefore\ A^c$ and B^c are independent.

6. Problem : *Three boxes B_1, B_2 and B_3 contain balls with different colours as shown below :*

	White	*Black*	*Red*
B_1	2	1	2
B_2	3	2	4
B_3	4	3	2

A die is thrown. B_1 is chosen if either 1 or 2 turns up. B_2 is chosen if 3 or 4 turns up and B_3 is chosen if 5 or 6 turns up. Having chosen a box in this way, a ball is chosen at random from this box. If the ball drawn is found to be red, find the probability that it is drawn from box B_2.

Solution : Let E_i be the event of choosing the box B_i and $P(E_i)$ be the probability of choosing the box B_i, $i = 1, 2, 3$. Then

$$P(E_1) = P(E_2) = P(E_3) = \frac{2}{6} = \frac{1}{3}$$

Having chosen the box B_i, the probability of drawing a red ball, $P(R|E_i)$, is given by $P(R|E_1) = \frac{2}{5}$, $P(R|E_2) = \frac{4}{9}$, and $P(R|E_3) = \frac{2}{9}$. We have to find the probability $P(E_2|R)$.

By Baye's theorem $$P(E_2 | R) = \frac{P(E_2)\,P(R|E_2)}{P(E_1)\,P(R|E_1) + P(E_2)\,P(R|E_2) + P(E_3)\,P(R|E_3)}$$

$$= \frac{\frac{1}{3}\times\frac{4}{9}}{\frac{1}{3}\left(\frac{2}{5}+\frac{4}{9}+\frac{2}{9}\right)} = \frac{4/27}{\frac{18+20+10}{3\times5\times9}} = \frac{5}{12}$$

Note : In all the above problems the sample space is finite. In the following case, the sample space is countably infinite.

7. Problem : *An urn contains w white balls and b black balls. Two players Q and R alternatively draw a ball with replacement from the urn. The player that draws a white ball first wins the game. If Q begins the game, find the probability of his winning the game.*

Solution : Let W denote the event of drawing a white ball in any draw and B that of a black ball.

Then $P(W) = \frac{w}{w+b}$, $P(B) = \frac{b}{w+b}$. Let E be the event of Q winning the game first.

Now P (Q wins the game): P(E) $= P(W \cup BBW \cup BBBBW \cup \ldots)$

$$= P(W) + P(BBW) + P(BBBBW) + \ldots$$

$$= P(W) + P(B)\,P(B)\,P(W) + P(B)\,P(B)\,P(B)\,P(B)\,P(W) + \ldots$$

$$= P(W)\left[1 + (P(B))^2 + (P(B))^4 + \ldots\right]$$

$$= P(W).\frac{1}{1-(P(B))^2} = \frac{\frac{w}{w+b}}{1-\left(\frac{b}{w+b}\right)^2} = \frac{w+b}{w+2b}$$

Exercise 9(c)

I. 1. Three screws are drawn at random from a lot of 50 screws, 5 of which are defective. Find the probability of the event that all 3 screws are non-defective, assuming that the drawing is (a) with replacement (b) without replacement

2. If A, B, C are three independent events of an experiment such that ,

$$P(A \cap B^c \cap C^c) = \frac{1}{4},\ P(A^c \cap B \cap C^c) = \frac{1}{8},\ P(A^c \cap B^c \cap C^c) = \frac{1}{4},$$

then find P(A), P(B) and P(C)

3. There are 3 black and 4 white balls in one bag, 4 black and 3 white balls in the second bag. A die is rolled and the first bag is selected if the die shows up 1 or 3, and the second bag for the rest. Find the probability of drawing a black ball, from the bag thus selected.

4. A, B, C are aiming to shoot a balloon, A will succeed 4 times out of 5 attempts. The chance of B to shoot the balloon is 3 out of 4 and that of C is 2 out of 3. If the three aim the balloon simultaneously, then find the probability that atleast two of them hit the balloon.

5. If A, B are two events, then show that $P(A|B)\ P(B) + P(A|B^c)\ P(B^c) = P(A)$

6. A pair of dice is rolled. What is the probability that they sum to 7 given that neither die shows a 2 ?

7. A pair of dice is rolled. What is the probability that neither die shows a 2 given that they sum to 7.

8. If A, B are any two events in an experiment and $P(B) \neq 1$. Show that

$$P(A|B^c) = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

9. An urn contains 12 red balls and 12 green balls. Suppose two balls are drawn one after another without replacement. Find the probability that the second ball drawn is green, given that the first ball drawn is red.

10. A single die is rolled twice in succession. What is the probability that the number on the second toss is greater than that on the first rolling?

11. If one card is drawn at random from a pack of cards then show that the event of getting an ace and getting a heart are independent events.

12. The probability that a boy A will get a scholarship is 0.9 and that another boy B will get is 0.8. What is the probability that atleast one of them will get the scholarship?

13. If A, B are two events with $P(A \cup B) = 0.65$, $P(A \cap B) = 0.15$, then Find the value of $P(A^c) + P(B^c)$

14. If A, B, C are independent events, then show that $A \cup B$ and C are also independent events.

15. A, B are two independent events such that, the probability of both the events to occur is $\frac{1}{6}$ and the probability of both the events do not occur is $\frac{1}{3}$. Find P(A).

16. A fair die is rolled. Consider the events

    $A = \{1, 3, 5\}$, $B = \{2, 3\}$ and $C = \{2, 3, 4, 5\}$. Find

    (i) $P(A \cap B)$, $P(A \cup B)$ (ii) $P(A|B)$, $P(B|A)$

    (iii) $P(A|C)$, $P(C|A)$ (iv) $P(B|C)$, $P(C|B)$

17. If A, B, C are three events in a random experiment, prove the following

    (i) $P(A|A) = 1$

    (ii) $P(\phi|A) = 0$

    (iii) $A \subseteq B \Rightarrow P(A|C) \le P(B|C)$

    (iv) $P(A - B) = P(A) - P(A \cap B)$

    (v) If A, B are mutually exclusive and $P(B) > 0$ then $P(A|B) = 0$

    (vi) If A, B are mutually exclusive then $P(A|B^c) = \dfrac{P(A)}{1 - P(B)}$ when $P(B) \neq 1$

    (vii) If A and B are mutually exclusive and $P(A \cup B) \neq 0$, then

$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}$$

18. Suppose that a coin is tossed three times. Let event A be "getting three heads" and B be the event of "getting a head on the first toss". Show that A and B are dependent events.

19. Suppose that an unbiased pair of dice is rolled. Let A denote the event that the same number shows on each die. Let B denote the event that the sum is greater than 7. Find (i) $P\left(\frac{A}{B}\right)$ (ii) $P\left(\frac{B}{A}\right)$.

20. Prove that A and B are independent events if and only if $P\left(\frac{A}{B}\right)=P\left(\frac{A}{B^C}\right)$.

II. 1. Suppose A and B are independent events with $P(A) = 0.6$, $P(B) = 0.7$. Then compute (i) $P(A \cap B)$ (ii) $P(A \cup B)$ (iii) $P(B/A)$ (iv) $P(A^c \cap B^c)$

2. The probability that Australia wins a match against India in a cricket game is given to be $\frac{1}{3}$. If India and Australia play 3 matches, what is the probability that (i) Australia will loose all the three matches? (ii) Australia will win atleast one match?

3. Three boxes numbered I, II, III contain the balls as follows :

	White	Black	Red
I	1	2	3
II	2	1	1
III	4	5	3

One box is randomly selected and a ball is drawn from it. If the ball is red, then find the probability that it is from box II.

4. A person secures a job in a construction company in which the probability that the workers go on strike is 0.65 and the probability that the construction job will be completed on time if there is no strike is 0.80. If the probability that the construction job will be completed on time even if there is a strike is 0.32, determine the probability that the construction job will be completed on time.

5. For any two events A, B show that

$$P(A \cap B) - P(A)\,P(B) = P(A^C)\,P(B) - P(A^C \cap B) = P(A)\,P(B^C) - P(A \cap B^C)$$

III. 1. Three Urns have the following composition of balls

Urn I : 1 white , 2 black

Urn II : 2 white , 1 black

Urn III : 2 white , 2 black

One of the urns is selected at random and a ball is drawn. It turns out to be white. Find the probability that it came from urn III.

2. In a shooting test the probability of A, B, C hitting the targets are $\frac{1}{2}, \frac{2}{3}$ and $\frac{3}{4}$ respectively. If all of them fire at the same target, Find the probability that (i) only one of them hits the target, (ii) atleast one of them hits the target.

3. In a certain college, 25% of the boys and 10% of the girls are studying mathematics. The girls constitute 60% of the student strength. If a student selected at random is found studying mathematics, find the probability that the student is a girl.

4. A person is known to speak truth 2 out of 3 times. He throws a die and reports that it is 1. Find the probability that it is actually 1.

Key Concepts

- A random experiment is one in which
  (i) the experiment can be repeated any number of times under identical conditions,
  (ii) all possible outcomes of the experiment are known in advance, and
  (iii) the actual outcome in a particular case is not known in advance.
- A sample space is the set of all outcomes of a random experiment.
- By an event A, we mean a subset of the sample space.
- An event E is impossible or certain according as $E = \phi$ or $E = S$ respectively.
- The complementary event of an event E is denoted by $E^c = S - E$
- Events $E_1, E_2, ..., E_n$ are mutually exclusive if $E_i \cap E_j = \phi$ for $i \neq j, \ 1 \leq i, \ j \leq n$.

- Events $E_1, E_2, \ldots, E_n$ are equally likely if there is no reason to expect one of them to happen in preference to the other.
- $E_1, E_2, \ldots, E_n$ are called exhaustive events if $E_1 \cup E_2 \cup \ldots\ldots \cup E_n = S$
- Classical definition of probability : If a random experiment results in n exhaustive, mutually exclusive and equally likely cases and m out of them are favourable to the happening of an event E, then the probability of E, denoted by P(E) is defined by $P(E) = \frac{m}{n}$
- For any event E, $0 \le P(E) \le 1$
- Relative frequency definition of Probability : Suppose E is an event of a random experiment. Let the experiment be repeated n times out of which E occurs $m(n)$ times. Then the ratio $r_n = \frac{m(n)}{n}$ is called the n-th relative frequency of the event E. If there exists an l such that $\lim_{n \to \infty} r_n = l$, then l is called the probability of E.
- Kolmogorov's approach to probability function (P) : Suppose S is the sample space of a random experiment and S is finite. Then a function $P : \mathscr{P}(S) \to \mathbf{R}$ satisfying the following axioms is called a probability function :

  (i) $P(E) \ge 0 \ \forall E \in \mathscr{P}(S)$ (ii) $P(S) = 1$

  (iii) If $E_1, E_2 \in \mathscr{P}(S)$, $E_1 \cap E_2 = \phi$, then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

  For each $E \in \mathscr{P}(S)$, P(E) is called the probability of E. In cases where S is countably infinite then, axiom (iii) has to be modified as :

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

- $P(\phi) = 0$, $P(S) = 1$
- **Addition theorem on probability :** If E_1 and E_2 be any two events of a random experiment, then $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

❖ **Conditional Probability** of the occurrence of an event A, given that B has already happened is denoted by P(A|B) and is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \; P(B) \neq 0$$

❖ **Multiplication theorem on Probability** : If A and B are events of a sample space S and

$P(A) > 0$, $P(B) > 0$ then

$$P(A \cap B) = P(A)\, P(B|A) = P(B)\, P(A|B)$$

❖ **Independent events :** Events A and B of a sample space S are said to be independent if $P(A \cap B) = P(A)\, P(B)$. Otherwise they are said to be dependent.

❖ **Baye's theorem :** If $E_1, E_2, ..., E_n$ are mutually exclusive and exhaustive events of a random experiment with $P(E_i) > 0$ for $i = 1, 2, ..., n$ then

$$P(E_k|A) = \frac{P(E_k)\, P(A \mid E_k)}{\sum_{i=1}^{n} P(E_i)\, P(A|E_i)} \; ; \quad k = 1, 2, ..., n$$

Historical Note

Pierre - Simon Laplace (1749–1827) came from humble origins in Normandy in France. He had his early education in a school run by benedictines. He entered the University of Caen at 16 to study theology but soon realised that his real interests were in Mathematics. In 1769 he became a professor of mathematics at the Paris military school.

Laplace is best known for his contribution to celestial mechanics - the study of motion of heavenly bodies. His book "*Traite de mecanique celeste*" (Five volumes) is still considered a classic of 19th century. *Laplace* was one of the founders of probability theory. He defined the probability of an event as the ratio of the number of favourable outcomes to the total number of outcomes of an experiment. All his work in this area is documented in his book "*Theorie analytique des probabilites*" (1812). Being loyal to *Napoleon* and *King Louis* XVIII, he played a prominent role during the French revolution.

Answers

Exercise 9(a)

I. 1. (i) yes (ii) yes (iii) yes

Exercise 9(b)

I. 1. $\frac{3}{8}$ 2. (i) $\frac{1}{7}$ (ii) $\frac{6}{7}$ 3. $\left(\frac{5}{6}\right)^2$

4. $\frac{1}{4}, \frac{3}{13}, \frac{3}{52}, \frac{11}{26}$ 5. $\frac{7}{8}$ 7. $\frac{6}{11}, \frac{5}{11}$ 8. 35%

9. (i) $\frac{1}{3}$ (ii) $\frac{7}{15}$ 10. (i) $\frac{9}{19}$ (ii) $\frac{10}{19}$ 11. $\frac{3}{4}$

II. 1. $\left(\frac{35}{36}\right)^{24}$ 3. (i) $\frac{12}{143}$ (ii) $\frac{50}{143}$ (iii) $\frac{131}{143}$

4. No 5. $\frac{19}{45}$ 6. $\frac{2}{5}$

7. $\frac{4}{7}, \frac{2}{7}, \frac{1}{7}$ 8. (i) $\frac{^{12}C_3}{^{23}C_3}$ (ii) $\frac{^{4}C_3}{^{23}C_3}$ (iii) $\frac{12\times7\times4}{^{23}C_3}$

11. (i) $\frac{1}{2}$ (ii) $\frac{1}{6}$ 12. $\frac{4}{5}$

13. (i) $\frac{17}{33}$ (ii) $\frac{16}{33}$

Exercise 9(c)

I. 1. (a) $\left(\frac{9}{10}\right)^3$ (b) $\frac{1419}{1960}$ 2. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ 3. $\frac{11}{21}$

4. $\frac{5}{6}$ 6. $\frac{4}{25}$ 7. $\frac{2}{3}$ 9. $\frac{12}{23}$

10. $\frac{15}{36}$ 12. 0.98 13. 1.2 15. $\frac{1}{2}$ or $\frac{1}{3}$

16. (i) $\frac{1}{6}, \frac{2}{3}$ (ii) $\frac{1}{2}, \frac{1}{3}$ (iii) $\frac{2}{4}, \frac{2}{3}$ (iv) $\frac{1}{2}, 1$

19. (i) $\frac{1}{5}$ (ii) $\frac{1}{2}$

II. 1. (i) 0.42 (ii) 0.88 (iii) 0.7 (iv) 0.12

2. (i) $\frac{8}{27}$ (ii) $\frac{19}{27}$ 3. $\frac{1}{4}$ 4. 0.4880

III. 1. $\frac{1}{3}$ 2. $\frac{6}{24}, \frac{23}{24}$ 3. $\frac{3}{8}$ 4. $\frac{2}{7}$

Chapter 10

Random Variables and Probability Distributions

"Statistics is a method of learning from experience and decision making must have been practiced from the beginning of mankind"

- C.R. Rao

Introduction

Many problems of Science as well as daily life are concerned with a numerical value associated with the outcome of an experiment. For instance, we would like to know the probability that a coin shows up tails 11 times when it is tossed 20 times. To study problems of this type we introduce the concept of a random variable.

Suppose that an experiment leads to two possible outcomes - For example : when a coin is tossed, the possible outcomes are Head and Tail. If we conduct an experiment with two possible outcomes, then it is called a Bernoulli trail. In general a possible outcome of a Bernoulli trial is called either a success (S) or a failure (F).

If 'p' is the probability of success and 'q' is the probability of failure, then it follows that $p+q=1$. Many

Jakob Bernoulli
(1654 - 1705)

Jakob Bernoulli, *born at Basel (Switzerland), was a professor of mathematics at the university of Basel and continued there till his death in 1705. Bernoulli established several results in probability theory. This work includes the application of probability theory to games of chance. Jakob Bernoulli is best known for the work "Ars conjectandi".*

problems can be solved by determining the probability of k success in n independent Bernoulli trials with probability of success p and probability of failure $q = 1 - p$. Considered as a function of k, we call this function, the binomial distribution. Binomial distribution was discovered by Jakob Bernoulli (1654 - 1705).

In binomial distribution, we deal with a precisely known definite sample of size n. But there are situations in daily life and industry where this may not be possible. Moreover the event may be rare and casual. Also the successful number of outcomes in the sample space is very small and form a negligible proportion. The number of accidents in a factory, the number of snake bites reported in a habitat of a town/ locality, etc. are some instances to mention. In such cases we know the number of times an event occurs but not how many times it does not occur. In such cases binomial distribution is inapplicable. To explain such situations we employ Poisson distribution. It was derived in 1837 by the French mathematician and physicist, S.D. Poisson (1781-1840).

In this chapter, we shall define a random variable that quantifies the events of a sample space. We shall then introduce the concepts of mean and variance of a random variable. In the next section, a Bernoulli trial is explained and the binomial distribution is established. The mean and variance of this distribution are discussed later and the chapter concludes with the Poisson distribution, its significance and some of its applications.

10.1 Random Variables

As discussed in chapter 9, a random experiment is associated with a sample space S. Many problems are concerned with a numerical value associated with the outcome (W) of an experiment. For example in the random experiment of tossing two fair coins, we have the sample space S = {HH, HT, TH, TT}. Suppose to each of the four sample points in S, a number is assigned as indicated below :

Outcome(W)	HH	HT	TH	TT
Number	2	1	1	0

Here the assigned number indicates the number of heads in each outcome W. Let the number of heads be denoted by X. Then X is a function on the sample space taking values 0,1,2. We call such a function which quantifies the events of a sample space, a random variable.

10.1.1 Definition

Let S be the sample space associated with a random experiment.
A function $\mathrm{X} : \mathrm{S} \to \mathbf{R}$ *is called a random variable.*

Note : If X is a random variable then $X^{-1}(\mathscr{P}(\mathbf{R})) = \mathscr{P}(S)$

Here P Stands for the Probability function (ref.see. 9.2.8) and $\mathscr{P}(S)$ stands for the power set of S

10.1.2 Examples

1. Example : Let S be the sample space of the experiment of rolling a fair die.

Then $X : S \to \mathbf{R}$ given by $X(n) = 0$, if n is even
$= 1$, if n is odd

is a random variable. Here $S = \{1, 2, 3, 4, 5, 6\}$ and

$X(1) = X(3) = X(5) = 1$; $X(2) = X(4) = X(6) = 0$.

2. Example : Three coins are tossed simultaneously. Then S = {HHH, HTH, HHT, HTT, THH, THT, TTH, TTT} is the sample space of this experiment. Define $X : S \to \mathbf{R}$ as $X(a)$ = the number of heads that shows for each $a \in S$. Then X is a random variable taking the following values:

$$X(HHH) = 3$$
$$X(HHT) = X(HTH) = X(THH) = 2$$
$$X(TTH) = X(THT) = X(HTT) = 1$$
$$X(TTT) = 0$$

Moreover we observe here $X^{-1}(0) = \{TTT\}$,

$X^{-1}(1) = \{HTT, THT, TTH\}$,

$X^{-1}(2) = \{HHT, HTH, THH\}$,

$X^{-1}(3) = \{HHH\}$ are some events of the experiment.

Let S be the sample space of a random experiment and P be a probability function on $\mathscr{P}(S)$. Then any random variable X on S gives rise to a probability function on $\mathscr{P}(\mathbf{R})$. This can be seen from the following theorem, the proof of which is beyond the scope of this book.

10.1.3 Theorem

Suppose S *is the sample space of a random experiment,* $P : \mathscr{P}(S) \to \mathbf{R}$ *is a probability function and* $X : S \to \mathbf{R}$ *is a random variable. Then the function* $\mathbf{P'}$ *defined on* $\mathscr{P}(\mathbf{R})$ *by*

$P'(Y) = P(X^{-1}(Y))$ *for each* $Y \in \mathscr{P}(\mathbf{R})$ *is a probability function on* $\mathscr{P}(\mathbf{R})$. *Here* $\mathbf{P'}$ *is called the probability function induced by* X.

10.1.4 Examples

1. Example : Let S be the sample space of the random experiment of rolling a fair die. Define $X : S \to \mathbf{R}$ by

$$X(n) = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}$$

We find P'

Here $S = \{1, 2, 3, 4, 5, 6\}$, $X^{-1}(0) = \{n \in S : X(n) = 0\} = \{2,4,6\}$

$X^{-1}(1) = \{n \in S : X(n) = 1\} = \{1,3,5\}$

The range of X is $\{0,1\}$

$$P'(0) = P(X^{-1}(0)) = P(\{2,4,6\}) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6} = \frac{1}{2} \text{ and}$$

$$P'(1) = P(X^{-1}(1)) = P(\{1,3,5\}) = \frac{1}{2}$$

Hence for $Y \in \mathscr{P}(\mathbf{R})$, $P'(Y) = 0$, if $\{0,1\} \cap Y = \phi$

$$= \begin{cases} \frac{1}{2}, & \text{if } 0 \text{ or } 1 \in Y, \text{ but not both} \\ 1, & \text{if } \{0, 1\} \subseteq Y \end{cases}$$

2. Example : Let S be the sample space of the random experiment of tossing three fair coins simultaneously. Define $X : S \to \mathbf{R}$ by $X(x)$ = the number of heads in X for each $x \in S$. Then X is a random variable with range $E = \{0, 1, 2, 3\}$. Here

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

Now

$$X^{-1}(0) = \{x \in S : X(x) = 0\} = \{TTT\}$$
$$X^{-1}(1) = \{x \in S : X(x) = 1\} = \{THT, TTH, HTT\}$$
$$X^{-1}(2) = \{x \in S : X(x) = 2\} = \{HHT, HTH, THH\}$$
$$X^{-1}(3) = \{x \in S : X(x) = 3\} = \{HHH\}$$

Now

$$P'(0) = P(X^{-1}(0)) = P(\{TTT\}) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$$

$$P'(1) = P(X^{-1}(1)) = P(\{THT, TTH, HTT\}) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{3}{8}$$

$$P'(2) = P(X^{-1}(2)) = P(\{HHT, HTH, THH\}) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{3}{8}$$

$$P'(3) = P(X^{-1}(3)) = P(\{HHH\}) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$$

For any $Y \in \mathscr{P}(\mathbf{R})$ We can compute $P(Y)$ using the above information and the addition theorem of chapter 9.

Thus $P'(Y) = 0$ if $E \cap Y = \phi$

$$= \frac{1}{8}, \text{ if } E \cap Y = \{0\} \text{ or } \{3\}$$

$$= \frac{3}{8}, \text{ if } E \cap Y = \{1\} \text{ or } E \cap Y = \{2\}$$

Suppose S is the sample space of a random experiment. Let $X : S \to \mathbf{R}$ be a random variable. Then to each $x \in \mathbf{R}$, the event :

$$\{a \in S : X(a) \le x\} \text{ is denoted by } (X \le x)$$

$$\text{i.e., } (X \le x) = \{a \in S : X(a) \le x\}$$
$$= \{a \in S : X(a) \in (-\infty, x]\}$$
$$= X^{-1}((-\infty, x]), \quad \text{for each } x \in \mathbf{R}$$

10.1.5 Definition

Suppose X *is a random variable connected with a random experiment and* P, *a probability function associated with it. Then* $F : \mathbf{R} \to \mathbf{R}$ *defined by* $F(x) = P(X \le x)$ *for each* $x \in \mathbf{R}$, *is called the probability distribution function (p.d.f) of* X.

We now state some properties of the probability distribution function without proof through the following theorem:

10.1.6 Theorem : *Suppose* F *is the probability distribution function of* X.

Then (i) $0 \le F(x) \le 1 \; \forall \, x \in \mathbf{R}$

(ii) $x_1 \le x_2 \Rightarrow F(x_1) \le F(x_2)$

(iii) $\lim\limits_{x \to \infty} F(x) = 1$ and $\lim\limits_{x \to -\infty} F(x) = 0$

(iv) $\lim\limits_{t \to x_+} F(t) = F(x)$

10.1.7 Definition

Suppose $X : S \to \mathbf{R}$ *is a random variable. If the range of* X *is either finite or countably infinite, then* X *is called a* ***discrete random variable.***

A random variable which can take all real values in an interval (a, b) is called a ***continuous random variable.***

In all the examples we have discussed so far, the random variable is discrete. In this chapter we confine our discussion to the discrete random variables only.

Suppose X is a discrete random variable with range $E = \{x_i \mid i \ge 1\}$. E may be finite or countably infinite. With each possible outcome x_i, we associate a number $P(X = x_i) = P(x_i)$, called the probability of x_i. The number $P(x_i)$, $i = 1, 2, 3, \ldots$ must satisfy the following conditions :

(i) $P(x_i) \geq 0$ for every i (ii) $\sum_{i \geq 1} P(x_i) = 1$

The set $\{P(X = x_i) = P(x_i)\}$ is called the probability distribution of the discrete random variable X. The probability distribution of the discrete random variable X is given in the following table:

$X = x_i$	x_1	x_2	x_3	...	x_n
$P(X = x_i)$	$P(x_1)$	$P(x_2)$	$P(x_3)$	...	$P(x_n)$

10.1.8 Examples

1. Example : For the random experiment of tossing two coins simultaneously, the sample space $S = \{HH, HT, TH, TT\}$. For every x, define $X(x)$ as the number of heads in X. Then $X(x)$ is a random variable. Range of $X = \{0, 1, 2\}$.

Now $P(X = 0)$ = Probability of getting no heads = $P(\{TT\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(X = 1)$ = Probability of getting one head = $P(\{HT, TH\}) = \frac{1}{4} + \frac{1}{4} = \frac{2}{4}$

$P(X = 2)$ = Probability of getting two heads = $P(\{HH\}) = \frac{1}{4}$

The probability distribution of the random variable X is given in the following table:

$X = x_i$	0	1	2
$P(X = x_i)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

2. Example : For the random experiment of throwing a die, the probability distribution of $X (= x_i)$, the number on the face of the die, is given by the following table:

$X = x_i$	1	2	3	4	5	6
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The concept of a probability distribution is analogous to the frequency distribution in statistics. A frequency distribution tells us how the total frequency is distributed among different class intervals of the variable, whereas a probability distribution tells us how the total probability 1 is distributed among the values which the random variable can take or associated with.

10.1.9 Definitions (mean and variance of a random variable)

Suppose X *is a discrete random variable with range* $\{x_1, x_2, x_3, ..., x_i ...\}$ for $i = 1, 2, 3, ...$ *Suppose* $P(x = x_i)$ *is known. If the sum of the infinite series* $\sum_i x_i P(X = x_i)$ *is finite, we call it the mean of* X *and denote by* μ. *Similarly if* $\sum_i (x_i - \mu)^2 P(X = x_i)$ *is finite, then it is called the variance of* X.

The nonnegative square root of variance is called the standard deviation of X *and is denoted by* σ. *Therefore, by definition, we have*

$$\mu = \sum_i x_i \ P(X = x_i) \quad \text{and} \quad \sigma^2 = \sum_i (x_i - \mu)^2 P(X = x_i)$$

10.1.10 Result

$$\sigma^2 = \sum_i x_i^2 \ P(X = x_i) - \mu^2$$

Proof : Consider
$$\sigma^2 = \sum_i (x_i - \mu)^2 \ P(X = x_i)$$
$$= \sum_i (x_i^2 - 2\mu x_i + \mu^2) P(X = x_i)$$
$$= \sum_i x_i^2 \ P(X = x_i) - 2\mu \sum_i x_i P(X = x_i) + \mu^2 \sum_i P(X = x_i)$$
$$= \sum_i x_i^2 P(X = x_i) - 2\mu.\mu + \mu^2.1$$
$$= \sum_i x_i^2 P(X = x_i) - \mu^2$$

10.1.11 Solved Problems

1. Problem : *A cubical die is thrown. Find the mean and variance of* X, *giving the number on the face that shows up.*

Solution : Let S be the sample space and X be the random variable associated with S, where P(X) is given by the following table

$X = x_i$	1	2	3	4	5	6
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Mean of $X = \mu = \sum_{i=1}^{6} X_i P(X = x_i)$

$$= 1.\frac{1}{6} + 2.\frac{1}{6} + 3.\frac{1}{6} + 4.\frac{1}{6} + 5.\frac{1}{6} + 6.\frac{1}{6} = \frac{1}{6}\left(\frac{6\times 7}{2}\right) = \frac{7}{2}$$

Variance of $X = \sigma^2 = \sum_{i=1}^{6} x_i^2 P(X = x_i) - \mu^2$

$$= 1^2.\frac{1}{6} + 2^2.\frac{1}{6} + 3^2.\frac{1}{6} + 4^2.\frac{1}{6} + 5^2.\frac{1}{6} + 6^2.\frac{1}{6} - \left(\frac{7}{2}\right)^2$$

$$= \frac{1}{6}\left(\frac{6\times 7\times 13}{6}\right) - \frac{49}{4} = \frac{35}{12}$$

2. Problem : *The Probability distribution of a random variable X is given below:*

$X = x_i$	1	2	3	4	5
$P(X = x_i)$	k	$2k$	$3k$	$4k$	$5k$

Find the value of k and the mean and variance of X.

Solution : We have $\sum_{i=1}^{5} P(X = x_i) = 1$

$$\Rightarrow k + 2k + 3k + 4k + 5k = 1. \quad \therefore k = \frac{1}{15}$$

The mean μ of $X = \sum_{i=1}^{5} x_i P(X = x_i)$

$$= 1(k) + 2(2k) + 3(3k) + 4(4k) + 5(5k)$$

$$= k(1 + 4 + 9 + 16 + 25)$$

$$= \frac{1}{15}\times 55 = \frac{55}{15} = \frac{11}{3}$$

Variance σ^2 of $X = \Sigma\, x_i^2\, P(X=x_i) - \mu^2$

$$= 1.(k) + 4.(2k) + 9.(3k) + 16.(4k) + 25.(5k) - \left(\frac{11}{3}\right)^2$$

$$= k(1 + 8 + 27 + 64 + 125) - \left(\frac{11}{3}\right)^2$$

$$= \frac{1}{15}(225) - \left(\frac{11}{3}\right)^2 = 15 - \frac{121}{9} = \frac{14}{9}$$

3. Problem : *If x is a random variable with probability distribution*

$$P(X=k) = \frac{(k+1)c}{2^k},\ k = 0, 1, 2, \ldots \quad \textit{then find } c.$$

Solution : Since $P(X=k) = \dfrac{(k+1)c}{2^k}$, $k = 0, 1, 2, \ldots$ is the probability distribution of x,

we have, $\sum_{k=0}^{\infty} P(X=k) = 1$ i.e., $\sum_{k=0}^{\infty} \dfrac{(k+1)}{2^k}\, c = 1$

or $c \sum_{k=0}^{\infty} \dfrac{(k+1)}{2^k} = 1$

But $\sum_{k=0}^{\infty} \dfrac{(k+1)}{2^k} = 1 + \dfrac{2}{2} + \dfrac{3}{2^2} + \dfrac{4}{2^3} + \dfrac{5}{2^4} + \ldots = 4,$

Hence $4c = 1$ and $c = \dfrac{1}{4}$.

4. Problem : *Let X be a random variable such that*

$$P(X=-2) = P(X=-1) = P(X=2) = P(X=1) = \frac{1}{6} \text{ and } P(X=0) = \frac{1}{3}.$$

Find the mean and variance of X

Solution : The mean $\mu = \sum_{k=-2}^{2} P(X=k)$

$$= (-2)\times\frac{1}{6} + (-1)\times\frac{1}{6} + (0)\times\frac{1}{3} + (1)\times\frac{1}{6} + (2)\times\frac{1}{6} = 0$$

Now $\sigma^2 = \sum_{k=-2}^{2} (k-\mu)^2 P(X=k) = \sum_{k=-2}^{2} k^2 P(X=k)$

$$= (-2)^2\times\frac{1}{6} + (-1)^2\times\frac{1}{6} + (0)\times\frac{1}{3} + (1)^2\times\frac{1}{6} + (2)^2\times\frac{1}{6} = \frac{5}{3}$$

5. Problem : *Two dice are rolled at random. Find the probability distribution of the sum of the numbers on them. Find the mean of the random variable.*

Solution : When two dice are rolled, the sample space S consists of $6 \times 6 = 36$ sample points :

S = { (1,1), (1,2), ..., (1,6), (2,1), ..., (2,6), ..., (6,6)}. Let X denote the sum of the numbers on the two dice. Then the range of X = {2, 3, 4, ..., 12}

The probability distribution for X is given hereunder :

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$P(X = x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mean of $X = \mu = \sum_{i=2}^{12} x_i P(X = x_i)$

$$= 2.\frac{1}{36} + 3.\frac{2}{36} + 4.\frac{3}{36} + 5.\frac{4}{36} + ... + 12.\frac{1}{36}$$

$$= \frac{1}{36}(2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12)$$

$$= \frac{252}{36} = 7$$

Exercise 10(a)

I. 1. A *p.d.f.* of a discrete random variable is zero except at the points $x = 0,1,2$. At these points it has the value $P(0) = 3c^3$, $P(1) = 4c - 10c^2$, $P(2) = 5c - 1$ for some $c > 0$. Find the value of c.

2. Find the constant c , so that $F(x) = c\left(\frac{2}{3}\right)^x$, $x = 1, 2, 3, ...$

is the *p.d.f.* of a discrete random variable X.

3.

$X = x$	-2	-1	0	1	2	3
$P(X = x)$	0.1	k	0.2	$2k$	0.3	k

is the probability distribution of a random variable X . Find the value of k and the variance of X.

4.

$X = x$	-3	-2	-1	0	1	2	3
$P(X = x)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

is the probability distribution of a random variable X. Find the variance of X.

5. A random variable X has the following probability distribution.

$X = x$	0	1	2	3	4	5	6	7
$P(X = x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find (i) k (ii) the mean and (iii) $P(0<X<5)$

II. 1. The range of a random variable X is $\{0,1,2\}$. Given that

$P(X=0)=3c^3$, $P(X=1)=4c-10c^2$, $P(X=2)=5c-1$

(i) Find the value of c (ii) $P(X<1)$, $P(1<X\leq 2)$ and $(P(0<X\leq 3)$

2. The range of a random variable X is $\{1, 2, 3, ...\}$ and $P(X=k)=\frac{c^k}{k!}$;
($k = 1, 2, 3, ...$)

Find the value of c and $P(0<X<3)$

10.2 Theoretical discrete distributions - Binomial and Poisson distributions

Suppose that an experiment has only two possible outcomes. For instance, when a coin is tossed, the possible outcomes are head and tail. Each performance of an experiment with two possible outcomes is a Bernoulli trial. The two possible outcomes are termed as success and failure. If p is taken as the probability of success and q is the probability of failure, it follows that $p + q = 1$. Many problems can be solved by determining the probability of x successes when an experiment consists of n independent Bernoulli trials.

We shall discuss two theoretical frequency distributions in this section. Both these distributions deal with discrete random variable and the variables in these distributions are distributed according to some definite probability law which can be expressed mathematically. We first discuss the Binomial distribution, which was discovered by Jakob Bernoulli in the year 1700 and was published posthumously in 1713. We take up the Poisson distribution later.

10.2.1 Binomial distribution

Let a Bernoulli trial be conducted n number of times and let the occurrence of an event E in a trail be called a success and its non-occurrence, a failure. In a single trial let p be the probability of success and q be the probability of failure, so that $p+q=1$. Consider a finite number n of independent Bernoulli trails in which the probability p of success is the same for each trial. The number of successes in n-trials may be 0, 1, 2, ..., n and is obviously a random variable X.

We now find the probability of x successes and $(n-x)$ failures in n independent trials i.e., $P(X=x)$.

By multiplication theorem of probability, the probability of x successes and $(n-x)$ failures in 'n' trials is $\{p\times p\times...\times p....x \text{ times}\}\times\{q\times q\times...\times q....(n-x)\text{ times}\}$ i.e., $p^x\, q^{n-x}$.

But x successes and $(n-x)$ failures in n trials can take place in ${}^nC_x$ ways.

Hence $P(X=x) = {}^nC_x\, p^x\, q^{n-x} = {}^nC_x \,.\, q^{n-x} \,.\, p^x$

The distribution of X is summarised in the following table :

x	$P(X=x)$
0	${}^nC_0\, p^0\, q^{n-0}$
1	${}^nC_1\, p^1\, q^{n-1}$
2	${}^nC_2\, p^2\, q^{n-2}$
.	....
.	....
.	....
r	${}^nC_r\, p^r\, q^{n-r}$
.	....
.	....
n	${}^nC_n\, p^n\, q^0$

Clearly the successive terms for $P(X=x)$ where $x\in\{0,1,2,...,n\}$ are the successive terms of the binomial expansion for $(q+p)^n$.

Note that $\sum_{x=0}^{n} P(X=x)=1$, since $p+q=1$. Here n and p are called the parameters of the distribution.

10.2.2 Definition

A discrete random variable X is said to follow a binomial distribution (or simply a binomial variable with parameters n and p) where $0 < p < 1$ *if*

$$P(X = x) = {}^nC_x\, p^x\, q^{n-x}, \quad x \in \{0, 1, 2, ..., n\}$$

If X is a binomial variate with parameters n, p; then it is also described by writing $X \sim B(n, p)$

10.2.3 Note (i): Binomial distribution follows under the following experimental conditions :

1. The number n of trials is finite.
2. Each trial results in two mutually exclusive outcomes, termed as success and failure.
3. The n trials are independent of each other.
4. The probability p of success is constant for each trial.

Note (ii) : *If the number of trials in any one attempt is* n *and if there are N such sets (attempts), then the frequency function of the binomial distribution is given by*

$$N.P(x) = N.{}^nC_x\, p^x\, q^{n-x}, \quad x \in \{0, 1, 2, ..., n\}$$

The number of sets in which we get exactly x successes

$$= N.\ {}^nC_x\, p^x\, q^{n-x}, \ x \in \{0, 1, 2, ..., n\}$$

We now state the following theorem without proof :

10.2.4 Theorem

If $X \sim B(n, p)$, *then the mean* μ *and the variance* σ^2 *of* X *are equal to np and npq respectively.*

10.2.5 Poisson distribution

In binomial distribution, the number of trials n is precisely known and is finite. But there are some situations where this may not be possible. Also if the event is rare and casual, the successful events are quite a few in the sample space. If we know the number of times an event occurred but not how many times it did not occur, then the binomial distribution is inapplicable. In such cases we compute the binomial probabilities approximately by using Poisson distribution.

The following are some of the experiments which may be analysed using Poisson distribution :

(i) the number of telephone calls received at a telephone exchange in a given time interval.

(ii) the number of printing errors in a page of a book.

(iii) the number of road/rail accidents reported in a city in a given period of time (say a day)

(iv) the number of deaths due to snake bites in a particular locality in a year

(v) the number of defective blades in a packet of 100

10.2.6 Definition

In X is a discrete random variable that can assume values 0, 1, 2, 3,... such that for some fixed $\lambda > 0$,

$P(X=x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots,$ *then X is said to follow a Poisson distribution with parameter* λ and X *is called a Poisson random variable.*

The Poisson distribution is given by the following table

$X = x$	0	1	2	....	k	....
$P(X = x)$	$e^{-\lambda}$	$\dfrac{e^{-\lambda}\lambda}{1!}$	$\dfrac{e^{-\lambda}\lambda^2}{2!}$	....	$\dfrac{e^{-\lambda}\lambda^k}{k!}$	....

10.2.7 Note

Poisson distribution is a probability distribution,

since $P(X=x) > 0 \;\forall\, x$ and $\sum\limits_{x=0}^{\infty} P(X=x) = \sum\limits_{x=0}^{\infty} \dfrac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum\limits_{x=0}^{\infty} \dfrac{\lambda^x}{x!} = 1$

10.2.8 Note

Poisson distribution can be used under the following experimental conditions :

(i) Each trial results in two mutually exclusive outcomes, termed as success and failure.

(ii) the number n of such trials is sufficiently large.

(iii) the trials are independent of each other.

(iv) the probability of a success 'p' is very very small.

10.2.9 Poisson distribution as a limiting form of binomial distribution

Poisson distribution can be derived as the "limiting case" of binomial distribution in the following sense.

If $\lambda > 0$ for each positive integer $n > \lambda$, let X_n be the binomial random variable $B\left(n, \frac{\lambda}{n}\right)$. Using the fact that $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$ we can prove that for every non negative integer k,

$$\lim_{n\to\infty} P\left(X_n = k\right) = e^{-\lambda}\frac{\lambda^k}{k!}$$

10.2.10 Note

We can show that if X is a Poisson random variable with parameter λ, then the mean and variance of X are equal and each is equal to λ.

The mean of Poisson distribution with parameter $\lambda > 0$ is

$$\sum_{k=0}^{\infty} k.\frac{\lambda^k}{k!}e^{-\lambda} = \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{(k-1)}}{(k-1)!} = \lambda e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = \lambda e^{-\lambda}e^{\lambda} = \lambda$$

To find the variance we consider $\sum_{k=0}^{\infty} k^2\frac{\lambda^k}{k!}$

$$\text{This series} = \sum_{k=1}^{\infty} k\frac{\lambda^k}{(k-1)!}$$

$$= \sum_{k=1}^{\infty}(k\text{-}1\text{+}1)\frac{\lambda^k}{(k-1)!}$$

$$= \sum_{k=2}^{\infty}\frac{\lambda^k}{(k-2)!} + \sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!}$$

$$= \lambda^2\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} + \lambda\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$$

$$= (\lambda^2 + \lambda)\, e^{\lambda}$$

$$\text{Now the variance} = \sum_{k=1}^{\infty} k^2\frac{\lambda^k}{k!}e^{-\lambda} - \left(\sum_{k=1}^{\infty} k\frac{\lambda^k}{k!}e^{-\lambda}\right)^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

10.2.11 Solved Problems

1. Problem : 8 *coins are tossed simultaneously. Find the probability of getting atleast* 6 *heads.*

Solution : In the experiment of tossing a coin, the probability of getting a head $= \frac{1}{2}$ and the probability of getting a tail $= \frac{1}{2}$. The probability of getting r heads in a random throw of 8 coins is

$$P(X=r) = {}^8C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{8-r} = {}^8C_r\left(\frac{1}{2}\right)^8, \quad r = 0,1,2,\ldots 8$$

The Probability of getting atleast 6 heads is

$$P(X \geq 6) = P(X=6) + P(X=7) + P(X=8)$$

$$= \left(\frac{1}{2}\right)^8\left({}^8C_6 + {}^8C_7 + {}^8C_8\right) = \frac{37}{256}$$

2. Problem : *The mean and variance of a binomial distribution are 4 and 3 respectively. Fix the distribution and find* $P(X \geq 1)$

Solution : Here $X \sim B(n, p)$ is specified by $n\,p = 4 = \mu$ and $n\,p\,q = \sigma^2 = 3$

hence, $\frac{npq}{np} = q = \frac{3}{4}$, so that $p = \frac{1}{4}$. *Also* $np = 4$ and $p = \frac{1}{4} \Rightarrow n = 16$

Hence the binomial distribution is given by $p = \frac{1}{4}$, $q = \frac{3}{4}$ and $n = 16$

Now $P(X \geq 1) = 1 - P(X=0) = 1 - q^n = 1 - \left(\frac{3}{4}\right)^n = 1 - \left(\frac{3}{4}\right)^{16}$

3. Problem : *The probability that a person chosen at random is left handed (in hand writing) is* 0.1. *What is the probability that in a group of* 10 *people, there is one who is left handed.*

Solution : Here $n = 10$, find $p = \frac{1}{10} = 0.1$. Hence $q = 0.9$

We have to find $P(X = 1)$: the probability that exactly one out of 10 is left handed

$$P(X=1) = {}^{10}C_1\, p^1\, q^{10-1} = 10 \times 0.1 \times (0.9)^9 = (0.9)^9$$

4. Problem : *In a book of 450 pages, there are 400 typographical errors. Assuming that the number of errors per page follow the Poisson law, find the probability that a random sample of 5 pages will contain no typographical error.*

Solution : The average number of errors per page in the book is $\lambda = \dfrac{400}{450} = \dfrac{8}{9}$

The probability that there are r errors per page : $P(X=r) = \dfrac{e^{-\lambda}\lambda^r}{r!} = \dfrac{e^{-8/9}\left(\frac{8}{9}\right)^r}{r!}$

Hence $P(X=0) = e^{-8/9}$

The required probability that a random sample of 5 pages will contain no error is

$$[P(X=0)]^5 = \left(e^{-8/9}\right)^5.$$

5. Problem : *Deficiency of red cells in the blood cells is determined by examining a specimen of blood under a microscope. Suppose a small fixed volume contains on an average 20 red cells for normal persons. Using the Poisson distribution, find the probability that a specimen of blood taken from a normal person will contain less than 15 red cells.*

Solution : Here $\lambda = 20$. Let $P(X=r)$ denote the probability that a specimen taken from a normal person will contain r red cells. Then we have $P(X<15) = \sum_{r=0}^{14} P(X=r) = \sum_{r=0}^{14} \dfrac{e^{-20}}{r!}(20)^r$

6. Problem : *A Poisson variable satisfies* $P(X=1) = P(X=2)$. *Find* $P(X=5)$

Solution : We have $P(X=r) = \dfrac{e^{-\lambda}\lambda^r}{r!}$, $\lambda > 0$

Given that $P(X=1) = P(X=2)$,

$$\frac{\lambda e^{-\lambda}}{1!} = \frac{\lambda^2 e^{-\lambda}}{2!} \quad \text{i.e.,} \quad \lambda^2 = 2\lambda \Rightarrow \lambda(\lambda-2) = 0$$

$\therefore \lambda = 2$ (Since $\lambda > 0$). Hence $P(X=5) = \dfrac{e^{-2}\,2^5}{5!}$.

Exercise 10(b)

I. 1. In the experiment of tossing a coin n times, if the variable X denotes the number of heads and $P(X = 4), P(X = 5), P(X = 6)$ are in arithmetic progression then find n.

2. Find the minimum number of times a fair coin must be tossed so that the probability of getting at least one head is at least 0.8.

3. The probability of a bomb hitting a bridge is $\frac{1}{2}$ and three direct hits (not necessarily consecutive) are needed to destroy it. Find the minimum number of bombs required so that the probability of the bridge being destroyed is greater than 0.9.

4. If the difference between the mean and the variance of a binomial variate is $\frac{5}{9}$ then, find the probability for the event of 2 successes when the experiment is conducted 5 times.

5. One in 9 ships is likely to be wrecked, when they are set on sail, when 6 ships are on sail find the probability for (i) Atleast one will arrive safely. (ii) Exactly three will arrive safely

6. If the mean and variance of a binomial variable X are 2.4 and 1.44 respectively, find $P(1 < X \le 4)$.

7. It is given that 10% of the electric bulbs manufactured by a company are defective. In a sample of 20 bulbs , find the probability that more than 2 are defective.

8. On an average, rain falls on 12 days in every 30 days, find the probability that, rain will fall on just 3 days of a given week.

9. For a binomial distribution with mean 6 and variance 2, find the first two terms of the distribution.

10. In a city 10 accidents take place in a span of 50 days. Assuming that the number of accidents follows the Poisson distribution, find the probability that there will be 3 or more accidents in a day.

II. 1. Five coins are tossed 320 times. Find the frequencies of the distribution of number of heads and tabulate the result.

2. Find the probability of guessing at least 6 out of 10 of answers in (i) True or false type examination (ii) multiple choice with 4 possible answers.

3. The number of persons joining a cinema ticket counter in a minute has Poisson distribution with parameter 6. Find the probability that (i) no one joins the queue in a particular minute. (ii) two or more persons join the queue in a minute.

Key Concepts

- **Random variable :** If S is the sample space of a random experiment, then a function $X : S \to \mathbf{R}$ is called a random variable. The term random variable is a misnomer. It is neither random nor a variable.

- **Discrete random variable :** A random variable X whose range is either finite or countably infinite is called a discrete random variable. i.e., X is called a discrete random variable, if the range of X is either $\{x_1, x_2, \ldots, x_n\}$ *or* $\{x_1, x_2, \ldots\}$. Otherwise X is said to be a continuous random variable.

- **Probability function induced by a random variable :** Suppose S is the sample space of a random experiment. Let $P : \mathscr{P}(S) \to \mathbf{R}$ be a probability function and $X : S \to \mathbf{R}$, a random variable. Then $P' : \mathscr{P}(\mathbf{R}) \to \mathbf{R}$ defined by $P'(\mathbf{Y}) = P(X^{-1}(\mathbf{Y}))$ for each $Y \in P(\mathbf{R})$ is a probability function called the probability function induced by X.

  Probability distribution function : Suppose X is a random variable. Then $F : \mathbf{R} \to \mathbf{R}$ given by $F(x) = P(X \le x) = P(X^{-1}(-\infty, x])$ for each $x \in \mathbf{R}$ is called the probability distribution function of X.

- The probability distribution of a discrete random variable is given by

$X = x_i$	x_1	x_2	x_3	...	....	x_n	....
$P(X = x_i)$	p_1	p_2	p_3	...	.....	p_n	...

where $p_i \ge 0$ for $i = 1, 2, \ldots$ and $\sum_{i \ge 1} p_i = 1$.

- The mean (μ) and variance (σ^2) of a discrete random variable X are

$$\mu = \Sigma x_n P(X = x_n) \quad \text{and} \quad \sigma^2 = \Sigma (x_n - \mu)^2 P(X = x_n)$$
$$= \Sigma\left(x_n^2 P(X = x_n)\right) - \mu^2$$

The standard deviation σ is the non negative square root of the variance.

- If p is the probability of a success, q be the probability of a failure such that $p+q=1$ and n is the number of Bernoulli trials, then the probability distribution of a discrete random variable X, called a binomial variate is given by

$$P(X=k) = {}^{n}c_{k}\ p^{k} q^{n-k}, \quad k=0,1,\ 2,\ldots,n.$$

  This is called the binomial distribution.

  Here n and p are called the parameters of the distribution.

  In this case X is expressed as $X \sim B(n,\ p)$

- If X is a binomial variate with parameters n and p i.e., $X \sim B(n, p)$, then the mean of the distribution $\mu = n\,p$ and the variance of the distribution $\sigma^2 = n\,p\,q$. The standard deviation of this distribution is given by $\sqrt{n\,p\,q}$.

- The probability distribution of a discrete random variable X (called the Poisson variable) given by $P(X=k)=\dfrac{e^{-\lambda}\lambda^{k}}{k!}, k=0,1,2,\ldots$ and $\lambda>0$, is called the Poisson distribution.

  Here λ is called the parameter of X.

- If X is a Poisson variate with parameter λ then its mean μ = variance $\sigma^2=\lambda$.

- Poisson distribution can be approximated as a limiting case of binomial distribution under the following conditions:

  (i) the number of trials must be indefinitely large i.e., $n \to \infty$.

  (ii) p, the constant probability of success in each trial is very very small i.e., $p \to 0$

  (iii) $n.p=\lambda$ is a finite positive real number.

Historical Note

Random drawing of lots has been used throughout recorded history as an unbiased way to distribute risks or rewards. Randomness is in general a convenient way to allow local decisions to be made while maintaining overall averages. The notion of taking random samples and the concept of random variables have been common since 1900.

However the term random variable was first used by *Jakob Bernoulli*.

Answers

Exercise 10(a)

I. 1. $\frac{1}{3}$ 2. $\frac{1}{2}$

3. $k = 0.1, \quad \sigma^2 = 2.16$

4. $\sigma^2 = \frac{28}{9}$ 5. (i) $\frac{1}{10}$ (ii) 3.66 (iii) $\frac{4}{5}$

II. 1. (i) $\frac{1}{3}$ (ii) $\frac{1}{9}, \frac{2}{3}, \frac{8}{9}$

2. $\log_e 2;\ \log_e 2 + \frac{1}{2}(\log_e 2)^2$

Exercise 10(b)

I. 1. 7 or 14 2. 3 3. 9

4. $\frac{80}{243}$

5. (i) $1 - \frac{1}{9^6}$ (ii) $20\left(\frac{8^3}{9^6}\right)$ 6. $\frac{2268}{3125}$

7. $\sum_{k=3}^{20} {}^{20}C_k \dfrac{9^{20-k}}{10^{20}}$

8. $\dfrac{35\times 2^3 \times 3^4}{5^7}$

9. $n=9;\ \left(\dfrac{1}{3}\right)^9,\ \dfrac{2}{3^7}$

10. $\sum_{k=3}^{\infty} e^{-\lambda}\dfrac{\lambda^k}{\underline{|k}},\ \lambda=0.2$

II. 1.

n(H)	0	1	2	3	4	5
f	10	50	100	100	50	10

2. (i) $\sum_{k=6}^{10} {}^{10}C_k \left(\dfrac{1}{2}\right)^{10}$ (ii) $\sum_{k=6}^{10} {}^{10}C_k \left(\dfrac{1}{4}\right)^{k}\left(\dfrac{3}{4}\right)^{10-k}$

3. $\lambda = 6$; (i) e^{-6} (ii) $1-e^{-6}-6e^{-6}$

Appendix

- ☞ Exponential and Logarithmic Series
- ☞ Linear Programming

No Question is to be set in the IPE, Mathematics - IIA from the topics mentioned above

Chapter 1

Exponential and Logarithmic Series

In this topic we will discuss the representation of certain functions by exponential and logarithmic series, which are in the form of infinite series. We will explain the way in which they can be used to simplify a given series.

1.1 Exponential Series

Leonhard Euler (1707-1783), the great swiss mathematician introduced the number e in his classic work on calculus in 1748. From then onwards the number e is useful in calculus in as much as the number π in the study of geometry.

Corresponding to a sequence $\{a_n\}$, we associate an 'infinite series' denoted by $a_1 + a_2 + ... + a_n + ...$ or $\sum_{n=1}^{\infty} a_n$. Here a_n is called the n th term of the series and $\{s_n\}$, where $S_n = a_1 + a_2 + ... + a_n$ is called the sequence of partial sums. If $\{s_n\}$ converges, we say that Σa_n converges and $\lim s_n$ as the sum of the series.

Theorem : If $x \in \mathbf{R}$, then the series $1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ...$ is convergent.

This series is called the exponential series and we write

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ... = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

In particular, when $x = 1$ we write $e = e'$ so that

$$e = 1 + 1 + \frac{1}{2!} + ... + \frac{1}{n!} + ... \qquad ...(1)$$

We shall now estimate the value of e and show that $2 < e < 3$.

Consider the following two series.

$$\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots+\frac{1}{n!}+\ldots \qquad \ldots (2)$$

and
$$\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots+\frac{1}{2^{n+1}}+\ldots \qquad \ldots (3)$$

Note that $\frac{1}{3!}=\frac{1}{6}$ and $\frac{1}{2^2}=\frac{1}{4}$, so that $\frac{1}{3!}<\frac{1}{2^2}$,

$\frac{1}{4!}=\frac{1}{24}$ and $\frac{1}{2^3}=\frac{1}{8}$ so that $\frac{1}{4!}<\frac{1}{2^3}$,

$\frac{1}{5!}=\frac{1}{120}$ and $\frac{1}{2^3}=\frac{1}{16}$ so that $\frac{1}{5!}<\frac{1}{2^4}$,

and so on. By induction, we can easily show that $\frac{1}{n!}<\frac{1}{2^{n-1}}$, for $n\geq 2$.

Also, since each term in series (2) is less than the corresponding term in series (3), we can write

$$\left(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots+\frac{1}{n!}+\ldots\right)<\left(\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^{n+1}}+\ldots\right)$$

Adding $\left(1+\frac{1}{1!}+\frac{1}{2!}\right)$ on both sides of this inequality, we get

$$e=\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots.+\frac{1}{n!}+\ldots\right)<\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots.+\frac{1}{2^{n+1}}+\ldots\right)$$

$$=1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots.+\frac{1}{2^{n+1}}+\ldots\right)=1+\frac{1}{\left(1-\frac{1}{2}\right)}=1+2=3.$$

Hence $e<3$. Also, we observe from series (1), that $e>2$. Therefore $2<e<3$.

Remark : For all real x, we have

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

i.e., $$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}.$$

Deduction : For $a > 0$, and x real, we know that

$$a^x = e^{x \log_e a}$$

Hence $$a^x = 1 + \frac{x(\log_e a)}{1!} + \frac{x^2(\log_e a)^2}{2!} + \ldots + \frac{x^n(\log_e a)^n}{n!} + \ldots$$

1.2 Solved Problems

1. Problem : Find the coefficient of x^2 in the expansion of e^{2x+3} as a series in powers of x.

Solution : Since $e^{2x+3} = e^3 \,.\, e^{2x}$

$$= e^3 \left[1 + \frac{(2x)}{1!} + \frac{(2x)^2}{2!} + \ldots\right]$$

the coefficient of x^2 in the expansion of e^{2x+3} is $e^3\left(\frac{2^2}{2!}\right) = 2e^3$.

Aliter: Replacing x by $2x+3$ in the exponential series (1), we get

$$e^{2x+3} = 1 + \frac{2x+3}{1!} + \frac{(2x+3)^2}{2!} + \ldots + \frac{(2x+3)^n}{n!} + \ldots \qquad \ldots(4)$$

since $$\frac{(2x+3)^n}{n!} = \frac{(3+2x)^n}{n!}$$

we can expand the R.H.S. of equation (5) using binomial .... (5)

theorem and write it as $\frac{1}{n!}\left[3^n + {}^nc_1 3^{n-1}(2x) + {}^nc_2 3^{n-2}.(2x)^2 + \ldots + (2x)^n\right]$

Here the coefficient of x^2 is $\left(\frac{{}^nc_2.3^{n-2}.2^2}{n!}\right)$.

Hence the coefficient of x^2 is the series on the R.H.S of the equation (4) is

$$\sum_{n=2}^{\infty} \frac{nc_2.3^{n-2}.2^2}{n!} = 2\sum_{n=2}^{\infty} \frac{n(n-1).3^{n-2}}{n!}$$

$$= 2.\sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!}$$

$$= 2.\left[1+\frac{3}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\ldots\right] = 2\,.\,e^3.$$

Hence the coefficient of x^2 in the expansion of e^{2x+3} is $2e^3$.

2. Problem : Show that $\dfrac{e+e^{-1}}{2}=1+\dfrac{1}{2!}+\dfrac{1}{4!}+\dfrac{1}{6!}+\ldots$ and $\dfrac{e-e^{-1}}{2}=1+\dfrac{1}{3!}+\dfrac{1}{5!}+\ldots$

Solution : We know that $e^x=1+x+\dfrac{x^2}{2!}+\ldots+\dfrac{x^n}{n!}+\ldots$

We already have $e=1+\dfrac{1}{1!}+\dfrac{1}{2!}+\ldots+\dfrac{1}{n!}+\ldots$

Putting $x=-1$, we have $e^{-1}=1-\dfrac{1}{1!}+\dfrac{1}{2!}-\dfrac{1}{3!}+\ldots+(-1)^n\dfrac{1}{n!}+\ldots$

Adding the above two equations we get

$$e+e^{-1}=2\left(1+\frac{1}{2!}+\frac{1}{4!}+\ldots\right)$$

or $$\frac{e+e^{-1}}{2}=1+\frac{1}{2!}+\frac{1}{4!}+\ldots$$

Again by subtracting the same equations we get

$$e-e^{-1}=2\left[\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\ldots\right]$$

or $$\frac{e-e^{-1}}{2}=\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\ldots$$

3. Problem : Show that $\frac{1.2}{1!}+\frac{2.3}{2!}+\frac{3.4}{3!}+...=3e$.

Solution : The n^{th} term (T_n) of the given series is

$$T_n=\frac{n(n+1)}{n!}=\frac{n+1}{(n-1)!}=\frac{(n-1)+2}{(n-1)!}$$

$$=\frac{n-1}{(n-1)!}+\frac{2}{(n-1)!}$$

$$\therefore \sum_{n=1}^{\infty}T_n=\sum_{n=1}^{\infty}\frac{n-1}{(n-1)!}+2\sum_{n=1}^{\infty}\frac{1}{(n-1)!}$$

$$=\sum_{n=2}^{\infty}\frac{1}{(n-2)!}+2\sum_{n=1}^{\infty}\frac{1}{(n-1)!}$$

$$=e+2e=3e.$$

4. Problem : Find the sum of the series $\frac{4}{1!}+\frac{11}{2!}+\frac{22}{3!}+\frac{37}{4!}+....$

Solution : The n-th term T_n of the given series

$$T_n=\frac{2n^2+n+1}{n!}=\frac{2n+1}{(n-1)!}+\frac{1}{n!}$$

$$=\frac{2n}{(n-1)!}+\frac{1}{(n-1)!}+\frac{1}{n!}$$

$$=\frac{(2n-2)+2}{(n-1)!}+\frac{1}{(n-1)!}+\frac{1}{n!}$$

$$=\frac{2(n-1)}{(n-1)!}+\frac{2}{(n-1)!}+\frac{1}{(n-1)!}+\frac{1}{n!}$$

$$\therefore \sum_{n=1}^{\infty}T_n=2\sum_{n=1}^{\infty}\frac{(n-1)}{(n-1)!}+2\sum_{n=1}^{\infty}\frac{1}{(n-1)!}+\sum_{n=1}^{\infty}\frac{1}{(n-1)!}+\sum_{n=1}^{\infty}\frac{1}{n!}$$

$$=2\sum_{n=2}^{\infty}\frac{1}{(n-2)!}+2e+e+(e-1)$$

$$=2e+2e+e+e-1=6e-1.$$

5. Problem : Find the coefficient of x^k in the expansion of $\left(\dfrac{1-ax-x^2}{e^x}\right)$.

Solution : $\dfrac{1-ax-x^2}{e^x} = (1-ax-x^2)\, e^{-x}$

$$= (1-ax-x^2)\left(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots+\frac{(-1)^k x^k}{k!}+\ldots\right)$$

The coefficient of x^k in the product on the R.H.S is

$$\frac{(-1)^k}{k!}-\frac{a(-1)^{k-1}}{(k-1)!}-\frac{(-1)^{k-2}}{(k-2)!} = \frac{(-1)^k}{k!}[1+ak-k(k-1)].$$

Exercise 1(a)

1. Find the sum of the following series.

   (i) $1+\dfrac{3}{1!}+\dfrac{5}{2!}+\dfrac{7}{3!}+\ldots$

   (ii) $1+\dfrac{2x}{1!}+\dfrac{3x^2}{2!}+\dfrac{4x^3}{3!}+\ldots$

   (iii) $\dfrac{1^2}{2!}+\dfrac{2^2}{3!}+\dfrac{3^2}{4!}+\dfrac{4^2}{5!}+\ldots$

2. Show that $\dfrac{2}{3!}+\dfrac{4}{5!}+\dfrac{6}{7!}+\ldots = \dfrac{1}{e}$.

3. Show that $1+\dfrac{2^2}{2!}+\dfrac{3^2}{3!}+\ldots = 2e$.

4. Show that $1+\dfrac{2^3}{2!}+\dfrac{3^3}{3!}+\dfrac{4^3}{4!}+\ldots = 5e$.

5. Find the coefficient of x^n in the expansion of $(3+2x)\, e^{3x}$.

6. Show that $\frac{1}{2!}+\frac{1+2}{3!}+\frac{1+2+2^2}{4!}+\ldots=\frac{1}{2}(e-1)^2$.

7. Find the coefficient of x^2 in the expansion of e^{3x+4}.

8. Find the coefficient of x^3 in the expansion of e^{2x+3}.

1.3 Logarithmic Series

Logarithmic series is another important series which is also in the form of an infinite series. We shall now state the following theorem (without proof) and illustrate some of its applications.

Theorem : If $|x|<1$, i.e., $-1<x<1$, then

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots \qquad \ldots(1)$$

The series on the R.H.S. of equation (1) is called the logarithmic series.

Note : The expansion of $\log_e(1+x)$ is valid for $x=1$.

Hence, by substituting $x=1$ in the expansion (1), we get

$$\log_e 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{(-1)^{n-1}}{n}+\ldots$$

The following formulae are related to logarithmic series.

1. For $-1<x\le 1$, $[\log_e(1+x)]^2=2\left\{\frac{x^2}{2}-\frac{1}{3}\left(1+\frac{1}{2}\right)x^3+\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^4-\ldots\right\}$.

2. For $-1<x\le 1$, we have $|-x|=|x|<1$. Then replacing x by $-x$ in logarithmic series,

   we have $\log_e(1-x)=-\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)$.

3. For $|x|<1$, $\log_e(1+x)-\log_e(1-x)=\log_e\left(\frac{1+x}{1-x}\right)=2\left\{x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right\}$.

4. $\log_e(1+x)+\log_e(1-x)=-2\left(\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\ldots\right)$, for $|x|<1$.

1.4 Solved Problems

1. Problem : Sum the series $\frac{1}{1.2}-\frac{1}{2.3}+\frac{1}{3.4}-\frac{1}{4.5}+\ldots$

Solution : $\frac{1}{1.2}-\frac{1}{2.3}+\frac{1}{3.4}-\frac{1}{4.5}+\ldots$

$$=\left(\frac{1}{1}-\frac{1}{2}\right)-\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)+\ldots$$

$$=1-\frac{1}{2}-\frac{1}{2}+\frac{1}{3}+\frac{1}{3}-\frac{1}{4}-\frac{1}{4}+\ldots$$

$$=1-2\left(\frac{1}{2}\right)+2\left(\frac{1}{3}\right)-2\left(\frac{1}{4}\right)+\ldots$$

$$=2\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right]-1=2\log_e 2-\log_e e$$

$$=\log_e\left(\frac{4}{e}\right).$$

2. Problem : Sum the series $\frac{5}{1.2.3}+\frac{7}{3.4.5}+\frac{9}{5.6.7}+\ldots$

Solution : The n-th term (T_n) of the given series is

$$T_n=\frac{2n+3}{(2n-1)(2n)(2n+1)}=\frac{2}{2n-1}-\frac{3}{2n}+\frac{1}{2n+1}$$

$$=\frac{2}{2n-1}-\frac{2}{2n}-\frac{1}{2n}+\frac{1}{2n+1}$$

$$=2\left(\frac{1}{2n-1}-\frac{1}{2n}\right)+\left(-\frac{1}{2n}+\frac{1}{2n+1}\right)$$

$$\therefore \sum_{n=1}^{\infty}T_n=2\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)+\sum_{n=1}^{\infty}\left(-\frac{1}{2n}+\frac{1}{2n+1}\right)$$

$$=2\log_e 2+\log_e 2-1=3\log_e 2-1=\log_e 8-\log_e e=\log_e\left(\frac{8}{e}\right).$$

3. Problem : Show that $\log_e \sqrt{12} = 1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\ldots$

Solution : Consider the R.H.S. $= 1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\ldots$

$$= \left(\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{4^2}+\frac{1}{6}\cdot\frac{1}{4^3}+\ldots\right)+\left(1+\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{5}\cdot\frac{1}{4^2}+\frac{1}{7}\cdot\frac{1}{4^3}+\ldots\right)$$

$$= \left[\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2+\frac{1}{4}\left(\frac{1}{2}\right)^4+\frac{1}{6}\left(\frac{1}{2}\right)^6+\ldots\right]+\left[1+\frac{1}{3}\left(\frac{1}{2}\right)^2+\frac{1}{5}\left(\frac{1}{2}\right)^4+\frac{1}{7}\left(\frac{1}{2}\right)^6+\ldots\right]$$

$$= \left[\frac{1}{2}x^2+\frac{1}{4}x^4+\frac{1}{6}x^6+\ldots\right]+\left[1+\frac{1}{3}x^2+\frac{1}{5}x^4+\frac{1}{7}x^6+\ldots\right], \qquad (\text{taking } x=\frac{1}{2})$$

$$= \frac{1}{2}\left\{x^2+\frac{1}{2}x^4+\frac{1}{3}x^6+\ldots\right\}+\frac{1}{x}\left\{x+\frac{1}{3}x^3+\frac{1}{5}x^5+\ldots\right\}$$

$$= -\frac{1}{2}\log_e(1-x^2)+\frac{1}{2x}\log_e\left(\frac{1+x}{1-x}\right)$$

$$= -\frac{1}{2}\log_e\left(1-\frac{1}{4}\right)+\log_e\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right), \qquad \text{since } x=\frac{1}{2}$$

$$= -\frac{1}{2}\log_e\frac{3}{4}+\log_e 3 = -\frac{1}{2}\log_e\frac{3}{4}+\frac{1}{2}\log_e 9$$

$$= \frac{1}{2}\log_e\left(\frac{9\times 4}{3}\right)=\frac{\log_e 12}{2} = \log_e\sqrt{12}.$$

4. Problem : Sum the series : $\log_3 e-\log_9 e+\log_{27} e-\log_{81} e+\ldots$

Solution : We know that $\log_x a\,.\,\log_a x=1$ hence $\log_x a=\frac{1}{\log_a x}$.

Then $\log_3 e = \dfrac{1}{\log_e 3}$; $\log_9 e = \dfrac{1}{\log_e 3^2} = \dfrac{1}{2\log_e 3}$; $\log_{27} e = \dfrac{1}{\log_e 27} = \dfrac{1}{\log_e 3^3} = \dfrac{1}{3\log_e 3}$

$\log_{81} e = \dfrac{1}{\log_e 3^4} = \dfrac{1}{4\log_e 3}$, and so on.

Hence $\log_3 e - \log_9 e + \log_{27} e - \log_{81} e + \ldots = \dfrac{1}{\log_e 3} - \dfrac{1}{2\log_e 3} + \dfrac{1}{3\log_e 3} - \dfrac{1}{4\log_e 3} + \ldots$

$$= \frac{1}{\log_e 3}\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right] = \frac{\log_e 2}{\log_e 3}.$$

5. Problem : If $\log_e\left(\dfrac{1}{1-x-x^2+x^3}\right)$ is expanded in a series of ascending powers of x, then show that the coefficient of x^n will be $\dfrac{1}{n}$ or $\dfrac{3}{n}$ according as n is odd or even.

Solution : $1 - x - x^2 + x^3 = (1-x)(1-x^2)$

Hence $\log_e \dfrac{1}{1-x-x^2+x^3} = \log_e 1 - \log_e\{(1-x)(1-x^2)\}$

$$= -\log_e(1-x) - \log_e(1-x^2)$$

$$= \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\right) + \left(x^2 + \frac{x^4}{2} + \frac{x^6}{3} + \ldots\right)$$

If n is odd, the term containing x^n occurs only in the first series of R.H.S. and the coefficient of x^n is $\dfrac{1}{n}$. If n is even, the term containing x^n occurs in both the series of RHS. If $n = 2r$, then the coefficient of x^{2r} in the first series is $\dfrac{1}{2r}$ and that in the second series is $\dfrac{1}{r}$. Hence coefficient of x^{2r} is $\dfrac{1}{2r} + \dfrac{1}{r} = \dfrac{3}{2r}$.

Hence when n is even, coefficient of x^n is $\dfrac{3}{n}$.

Exercise 1(b)

1. Find the sum of the infinite sereis

(i) $\frac{1}{2}-\frac{1}{2.2^2}+\frac{1}{3.2^3}-\frac{1}{4.2^4}+\ldots$

(ii) $\frac{1}{1.3}-\frac{1}{2}\left(\frac{1}{3.5}\right)+\frac{1}{3}\left(\frac{1}{5.7}\right)+\ldots$

(iii) $\frac{1}{5}+\frac{1}{2.5^2}+\frac{1}{3.5^3}+\ldots$

2. Show that $\frac{1}{2.3}+\frac{1}{4.5}+\frac{1}{6.7}+\ldots=1-\log_e 2.$

3. Show that $\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\ldots.=\frac{1}{2}\log_e 2-\frac{1}{2}.$

4. Show that $\frac{1}{2}+\frac{3}{2}.\frac{1}{4}+\frac{5}{3}.\frac{1}{8}+\frac{7}{4}.\frac{1}{16}+\ldots=2-\log_e 2.$

5. Show that $\log_e 3=1+\frac{1}{3.2^2}+\frac{1}{5.2^4}+\frac{1}{7.2^6}+\ldots$

6. If $y = x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots,$ then show that $x = y-\frac{y^2}{2!}+\frac{y^3}{3!}-\frac{y^4}{4!}+\ldots.$

7. Show that $1+\frac{1}{3}\left(\frac{1}{2}\right)^2+\frac{1}{5}\left(\frac{1}{2}\right)^4+\frac{1}{7}\left(\frac{1}{2}\right)^6+\ldots=\log_e 3.$

8. Express $\log_e(1+3x+2x^2)$ in an infinite series in ascending powers of x. Show that the expansion is valid only if $|x|<\frac{1}{2}$.

9. Find the coefficient of x^8 and x^9 in the expansion of $\log_e(1+x+x^2)$ for $|x|<1$.

10. Find the sum of the series $\left(\frac{x-1}{x+1}\right)+\frac{1}{2}\frac{x^2-1}{(x+1)^2}+\frac{1}{3}\frac{x^3-1}{(x+1)^3}+\ldots.$

Answers

Exercise 1(a)

1. (i) $3e$ (ii) $(x+1)e^x$ (iii) $e-1$

5. $\dfrac{(-1)^n.3^{n-1}(9-2n)}{n!}$ 7. $\dfrac{9}{2}e^4$ 8. $\dfrac{4}{3}e^3$.

Exercise 1(b)

1. (i) $\log_e\left(\dfrac{3}{2}\right)$ (ii) $\log_e\left(\dfrac{4}{e}\right)$ (iii) $\log_e\left(\dfrac{5}{4}\right)$

9. $\dfrac{1}{8}, \dfrac{-2}{9}$ 10. $\log_e x$

Chapter 2

Linear Programming

Introduction

In the earlier classes you have learned how to solve a system of linear equations / inequations in two variables using a graphical method. Many problems of industry, business houses and daily life involve systems of inequations and / or equations having two or more variables. In this topic, we shall use our earlier knowledge in solving a few problems of the aforesaid nature involving only two variables, employing the graphical method. Problems involving more than two variables can be solved by analytical (algorithmic) methods and you will learn them in your higher classes.

We shall first consider a problem of real life situation to illustrate the nature of such problems and there after take up its mathematical formulation.

Consider a furniture dealer who sells tables and chairs. Let the cost Price of a table be ₹ 2500 and that of a chair be ₹ 500. He estimates that he can make a profit of ₹ 250 on selling each table and ₹ 75 on selling each chair. He has a total storage space that can accommodate 60 items in his show room and a total investment capacity of ₹ 50,000 on these two items of furniture. He has to findout how many tables and chairs he should buy with the available money and space inorder that he can maximize his profit.

Such type of problems in which we seek to maximize the profit or minimize the cost, time or resource form a general class of 'optimization problems'. Linear Programming Problems - which we are going to discuss here - form a subset or class of the optimization problems. In these problems, the optimizing function (profit function or cost / resource function) and the inequations/ equations involving the independent variables are all linear. *Linear Programming Problems (LPP) created interest and attained importance because of their wide applicability in Engineering, Economics, Commerce, Management etc.

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*[A mathematical expression of the form $c_1x_1 + c_2x_2 + ... + c_nx_n$, where $c_1, c_2, ..., c_n$ are real numbers and $x_1, x_2, ..., x_n$ are variables is called a linear form. The term 'Programming' refers to the process of determining a course or plan of action. In our case it is a method of finding the solution.]

In this topic, we shall study some linear Programming Problems, their mathematical formulation and solutions by graphical method.

2.1 Mathematical formulation of the LPP

We shall begin our discussion with the example of the furniture dealer and formulate this problem mathematically as a LPP.

The dealer can invest his total available money i.e., ₹ 50,000 exclusively on tables or chairs or combination of tables and chairs. If he decides to buy only tables, he can order for $50{,}000 \div 2500 = 20$ tables, over which he can make a profit of $20 \times 250 =$ ₹ 5000. If he chooses to buy only chairs, he can buy $50{,}000 \div 500 = 100$ chairs, but he is forced to buy only 60 chairs because of space limitation or constraint. In such case his profit is limited to $60 \times 75 =$ ₹ 4500. On the otherhand, if he chooses to buy tables and chairs, he can invest on various combinations. For example, if he chooses to buy 20 tables and 40 chairs, he gets a total profit of $(20 \times 250) + (40 \times 75) =$ ₹ 8000. Thus the dealer has several investment alternatives before him to earn profits. However, since his investment strategy or objective is to earn the maximum profit, we have to determine mathematically the exact number of tables and chairs he has to purchase, inorder that he can maximize his profit. To answer this question, we have to first formulate this problem mathematically as a LPP and then solve it.

Let x and y denote respectively the number of tables and chairs that the dealer buys with his available money. Then, assuming that he sells all these items, he earns a total profit of ₹ $250x + 75y$. His objective is to earn the maximum profit i.e., Max.Z. This can be expressed as a function of two variables x and y by

$$\text{Max. } Z = 250x + 75y \qquad \text{....(1)}$$

This is called the objective function of the LPP.

Since the dealer is constrained by his total investment capacity (₹ 50,000) and by the available storage space (60 items put together), we can express them mathematically by

$$2500x + 500y \leq 50{,}000$$

$$\text{(or)} \quad 5x + y \leq 100 \quad \text{(investment constraint)} \qquad \text{... (2)}$$

$$\text{and} \quad x + y \leq 60 \quad \text{(Storage space constraint)} \qquad \text{...(3)}$$

Also, the number of tables (x) and chairs (y) the dealer can buy is always non-negative. Hence we have

$$x \geq 0 \text{ and } y \geq 0 \quad \text{(non - negative restrictions)} \qquad \text{...(4)}$$

This is the mathematical formulation or version of the given problem.

In the problem considered above, the linear objective function Z, is to be maximized subject to two linear constraints (inequations (2) and (3)) and with non negative restrictions on decision variables x and y (inequations 4). But in a few other problems, we may have to minimize a linear objective function subject to certain other constraints i.e., inequalities of the $\geq$ type or equalities associated with non negative decision variables. All such problems are categorised broadly as 'Linear Programming Problems'.

Thus, more generally a LPP is one in which we have a linear objective function with a finite number of decision variables (say $x, y, z, \ldots$ etc) to be optimised (either maximize or minimize) subject to a finite set of linear constraints (in equalities and / or equalities) in terms of the decision variables, together with the non negative restrictions on the decision variables.

2.2 Different types of LPP

We shall enlist below a few LPP which are applicationally significant

(A) Diet Problem : In these problems we determine the quantity (in units) of food stuffs having specified vitamins which must be included in a diet so that the minimum daily requirement of these vitamins is met and that at the same time, the cost of purchasing these food stuffs is a minimum. (We assume here that taking more than the minimum requirement of vitamins is not harmful to the health of the consumer).

(B) Allocation Problem : Two crops (X and Y say) are to be grown on a given land of a Landlord. The per acre / hectare use of pesticide for the two crops are known and the maximum amount of pesticide that can be used for the two crops over the entire land is specified in the problem. Given the per acre/ hectare profit from these two crops, the problem is to determine the amount of available land to be allocated to each crop so as to maximize the total profit of the Landlord.

(C) Manufacturing Problem : In these problems, a manufacturer produces different products which are sold for a fixed profit per unit. Each product is to be processed on different machines which need a fixed machine time. If there is some limitation on the total machine time per day for each machine, the problem is to determine the number of units of the different products the manufacturers has to produce in order that he makes the maximum profit.

(D) Transportation Problem : In these problems we assume that there are two locations of factories manufacturing the same kind of product which is to be transported to different warehouses located at different regions. The amount of product produced (available) at each of the factories and the requirement of the commodity at each of the warehouses is known. Given the per unit cost of transporting the product from a factory location to a warehouse, the problem is to fix a transportation schedule i.e., the quantity of product to be transported from a factory location to a warehouse, so as to minimize the total cost of transportation of the available commodity to the requirement of the warehouses.

2.3 Graphical method for solving a LPP

Linear Programming Problems involving two decision variables can be easily solved by graphical method. Before we discuss the method, we shall recall a few essential concepts which you have studied in earlier classes. We confine ourselves to subsets of the plane.

1. **Convex set :** A line segment joining the points P and Q, denoted by $\overline{PQ}$, is the set of points that lie between P and Q including both.

   A subset X of a plane is said to be convex if $P, Q \in X \Rightarrow \overline{PQ} \subseteq X$. Geometrically, we say that X is convex if the line segment joining any two points $P, Q \in X$ is entirely contained in X.

Examples : The following sets are some examples of convex sets.

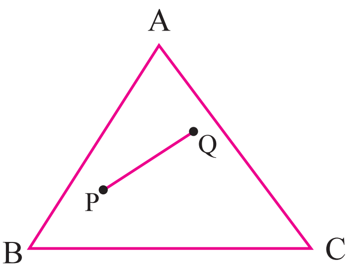

Fig. 2.1

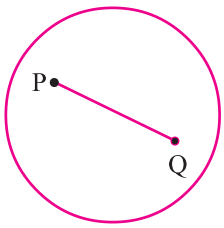

Fig. 2.2

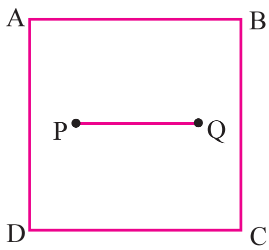

Fig. 2.3

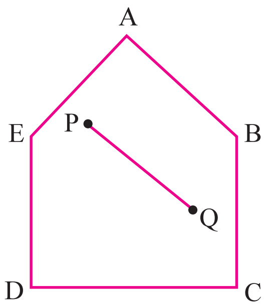

Fig. 2.4

The following are some examples of sets which are not convex.

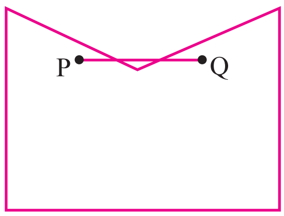

Fig. 2.5

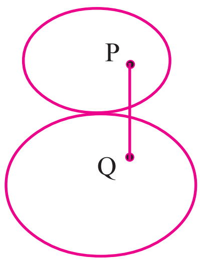

Fig. 2.6

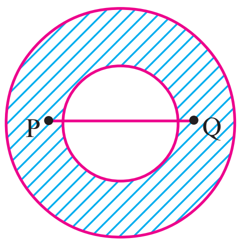

Fig. 2.7

Note : A convex region may be closed or may not be closed. It can also be an open region. However, a closed convex region is called a **convex polygon** if it is enclosed by a finite set of straight lines in a plane.

2. **Open convex region :** It is the set of all points within and on some region that is open on one side with a finite number of vertices.

3. **Extreme point of a convex region :** It is a corner point of the convex polygon enclosing the convex region.

   We shall now discuss the graphical method of solving a LPP, classified into the following two cases.

Case (i): All the constraint inequations / equations of the given LPP are of less than or equal to or equal to type and the solution set of the constraints is a closed convex polygon.

Step 1 : Considering each constraint inequality as an equality / equation, plot each constraint equation as a straight line in the plane (Since each linear equation in two variables represents a straight line in the plane)

Step 2 : Every point on a line thus drawn satisifes the equation of that line. Since each constraint is of less than or equal to type, the region lying the first quadrant below those lines is to be considered due to the non negative restrictions of the variables.

Step 3 : The points lying in the common region below the lines in the first quadrant satisfy all the constraints simultaneously. This common region is called the 'feasible region' of the given problem. Since all the constraints are of less than or equal to type, this feasible region is bounded. Every point of this region is called a feasible solution to the given problem and any point outside this region gives an infeasible solution of the problem.

Step 4 : Find the vertices of the corner (extreme) points of the closed convex polygon and evaluate the objective function at each of the vertices.

Step 5 : The coordinates of the vertex at which the objective function attains the maximum value (in the case of maximization type problems) or minimum value (in the case of minimization type problems) gives the desired optimum solution.

Case (ii) : One or more of the constraint inequations are of 'greater than or equal to' type i.e., when the solution set of the constraints is an open convex unbounded region.

In this case, we shall slightly modify the procedure and give a more general graphical method of solving a LPP.

Step 1 : Repeat the same procedure as given in Step 1 of case (i).

Step 2 : Observe that the feasible region is not bounded. Now choose a convenient value of z (say $z = 0$), the objective function, and plot the objective function line ($f(z) = 0$).

Step 3 : Pull this objective function line upto the extreme points (i.e., boundary Points) of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through atleast one corner point of the feasible region. In the minimization case, this line will stop at a point nearest to the origin and passing through at least one corner of the feasible region.

Step 4: Read the coordinates of the located extreme point(s) in Step 3. Find the maximum or minimum (as the case may be) value of z. This value of z is the desired optimum solution of the given LPP.

2.4 Solved Problems

1. Problem : Let us now solve the problem of investment in tables and chairs discussed earlier.

Solution: To solve this problem graphically, we shall consider the linear constraints and restrictions in its mathematical formulation.

$$5x + y \leq 100$$

$$x + y \leq 60$$

and $\quad x \geq 0,\ y \geq 0.$

Taking the constraint inequations as equations, we shall draw the lines representing these equations and consider only their segments lying in the first quadrant. The common region OABC determined by the two consitraints including the non negative restrictions $x, y \geq 0$ of the given LPP, which represents the feasible region, is shaded in the figure. Observe here that the feasible region is a closed convex polygon with the corner vertices O(0, 0), A(20, 0) B (10, 50), C (0, 60) (Fig. 2.8).

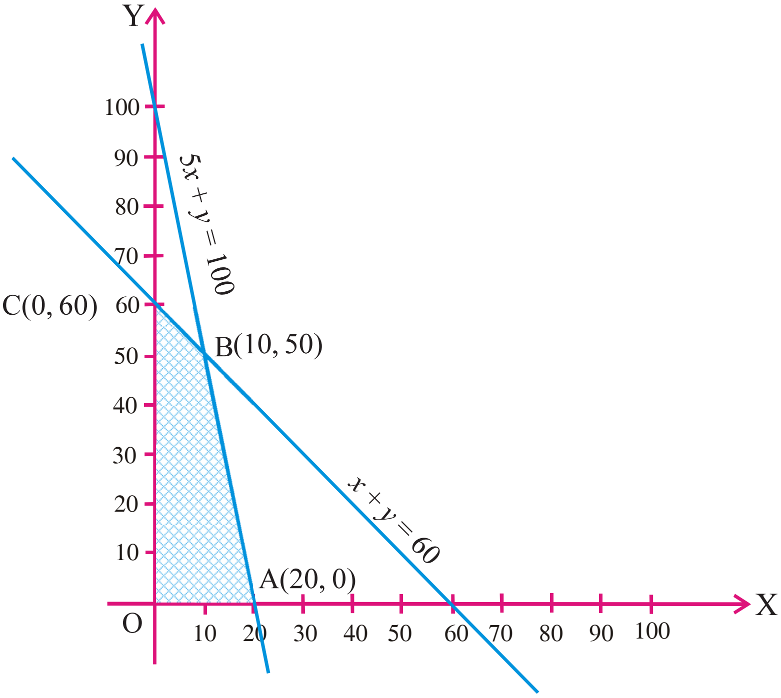

Fig. 2.8

Following the discussion made in case (i), the optimum solution for the given LPP exists at one of the vertices of this convex polygon OABC. Let us now compute the values of z at these points.

Vertex of the feasible region	Corresponding value of z (in ₹)
O(0, 0)	0
C(0, 60)	4,500
B(10, 50)	6,250 $\longrightarrow$ maximum
A(20, 0)	5,000

Observe that the maximum profit of ₹ 6,250 to the dealer results from the investment strategy (10, 50) i.e., $x = 10, y = 50$. Hence he has to buy 10 tables and 50 chairs.

2. Problem : We shall now solve the following diet problem.

The amount of vitamins V_1 and V_2 present in two different foods F_1 and F_2 respectively, the cost per kilogram of F_1, F_2 and the minimum daily requirement of the vitamins V_1 and V_2 are given in the following table. Assuming that consumption of more quantity of vitamins than the minimum requirement is not harmful, formulate and solve the following LPP so as to minimize the total cost of purchasing the mixture of foods F_1 and F_2.

Vitamin	Food Stuff		Minimum daily requirement (in units)
	F_1	F_2	
V_1	2	1	8
V_2	1	2	10
Cost of Foods(₹/kg.)	50	70	

Solution : We shall first formulate the problem as a LPP. Let x kg of food F_1 and y kg. of food F_2 make the mixture. Then the cost of purchasing these foods will be

$$z = 50x + 70y$$

Since this cost should be minimum, the objective function will be : Minimize $z = 50x + 70y$ ...(1)

This is subject to the constraints on the minimum daily requirement of vitamin V_1 and V_2 :

$$2x + y \geq 8 \qquad ...(2)$$

$$x + 2y \geq 10 \qquad ...(3)$$

and the non negative restrictions $x, y \geq 0$. ...(4)

Considering the constraint in equalities (2), (3) as equalities and drawing the straight lines for the linear equations, we obtain the Fig. 2.9. Observe here that the feasible region is not bounded i.e., an unbounded convex region.

The corner points of this open convex region are :

A(0, 8), B(2, 4) and C(10, 0).

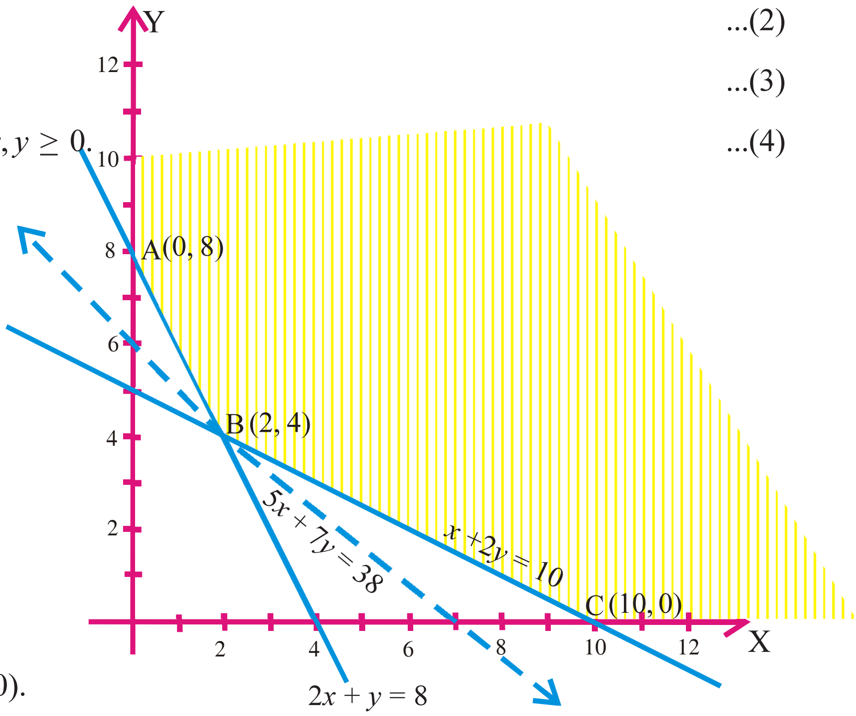

Fig. 2.9

We shall evaluate the objective function at these corner points.

Corner Point	Value of objective function $z = 50x + 70y$
A(0, 8)	560
B (2, 4)	380 $\longrightarrow$ minimum value
C(10, 0)	500

Since the feasible region is unbounded, following the discussion made in case (ii), to find the optimum solution, we have to consider the objective function straight line $50x + 70y = 0$ and pull this objective function line upto the extreme points or corners of the feasible region. Since this objective function is of minimization type, this line has to stop nearest to the origin and passing through one corner point. Observe that this line is $5x + 7y = 38$. We also observe that the graph of the inequality $5x + 7y < 38$ has no point in common with the feasible region.

Thus the minimum value of z is 380 and this is attained at the point B(2, 4). i.e., $x = 2, y = 4$. Hence the optimal mixing strategy would be to buy 2 kgs of food stuff F_1 and 4 kgs. of food stuff F_2. The total minimum cost is ₹ 380.

3. Problem : We shall now solve an allocation problem.

A cooperative society has 50 acres of land over which two crops A and B are to be cultivated. The per acre use of pesticide for the crops A and B are 20 litres and 10 litres respectively. The maximum amount of pesticide to be used over both the crops is restricted to 800 litres inorder to preserve the environment of the surroundings. If the per acre profit from the crops A and B are estimated as ₹ 1100 and ₹ 950 respectively, determine the land to be alloted to each crop so as to maximize the profit of the society.

Solution : Let x acres of land be alloted to crop A and y acres of land be alloted to crop B, so that $x \geq 0$ and $y \geq 0$.

Since the per acre profit of crop A is ₹ 1100 and per acre profit of crop B is ₹ 950, the total profit on both crops is ₹ $(1100x + 950y)$.

For the given L.P. Problem, the corresponding mathematical formulation is

Maximize $z = 1100\,x + 950\,y$

Subject to constraints

$x + y \leq 50$: (Land constraint)

$20x + 10y \leq 800$ i.e., $2x + y \leq 80$: (Pesticide constraint)

and $x \geq 0, \; y \geq 0$: (non negative restrictions)

To find the graphical solution, as before, we shall draw the straight lines representing the linear constriant equations and identify the feasible region OABC (Fig. 2.10). Since this region is a closed convex polygon, the corner points O(0, 0), A(40, 0), B(30, 20) and C(0, 50) are the extreme points.

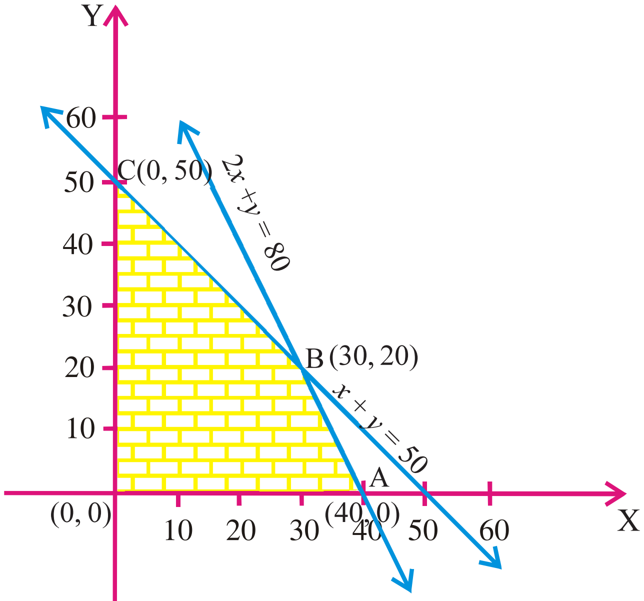

Fig. 2.10

We shall now evaluate the objective function at these corner points.

Corner Point	Value of objective function $z = 1100x + 950y$ (in ₹)
O(0, 0)	0
A (40, 0)	44,000
B (30, 20)	52,000 $\rightarrow$ maximum value
C (0, 50)	47,500

Since the maximum value of the objective function is attained at B(30, 20), we have $x = 30$, $y = 20$. Hence the society gets the maximum profit of ₹ 52,000 by allocating 30 acres for crop A and 20 acres for crop B.

4. Problem : We shall now solve a manufacturing problem.

A manufacturer makes two models A and B of a product. Each model has to be processed on two machines. Model A needs 1 hour of processing time on machine I and 2 hours of processing time on machine II to complete one unit. Model B needs 4 hours of processing time on machine I and 2 hours of processing time on machine II to complete one unit. Machine I works for a maximum of 8 hours per day and Machine II works for a maximum of 10 hours per day. If the per unit profit of model A is ₹ 200 and that of model B is ₹ 280, find the number of units of each model the manufacturer has to produce inorder to maximize his profit.

Solution : Let x be the number of units of model A and y be the number of units of model B to be manufactured per day. Then the profit function is $200x + 280y$. Hence the objective function is

Maximize $z = 200x + 280y$.

To make x units of model A and y units of model B, the machine I is to be used for $1.x + 4.y$ hours. But machine I can be used for not more than 8 hours per day.

$\therefore \quad x + 4y \le 8$: (time constraint on machine I)

Similarly to make x units of model A and y units of model B on machine II, it is to be used for $2x + 2y$ hours. But machine II cannot be used for more than 10 hours per day.

$\therefore \quad 2x + 2y \le 10$ or $x + y \le 5$: (time constraint on machine II)

Since x and y cannot be negative, $x \ge 0,\ y \ge 0$: (non negative restrictions)

The graph corresponding to the linear constraint equations and non negative restrictions is drawn below. We observe that OABC, the shaded region, is the closed convex polygon. This is the feasible region of the solution set of the given LPP with O(0, 0), A(5, 0), B(4, 1), C(0, 2) as corner points (Fig. 2.11).

We shall now evaluate the objective function at these corner points.

Corner Point	Value of the objective function $z = 200x + 280y$ (in ₹)
O(0, 0)	0
A (5, 0)	1,000
B (4, 1)	1,080 $\longrightarrow$ maximum value
C (0, 2)	560

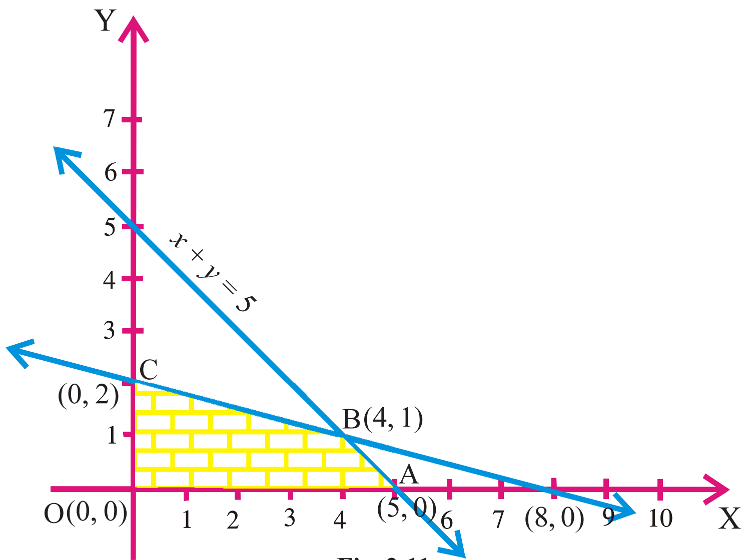

Fig. 2.11

Since the maximum profit is attained at B(4, 1), the manufacturer has to make 4 units of model A and 1 unit of model B per day to maximize his profit.

5. Problem : We shall now solve a transportation problem.

There are two factory locations P and Q of a company manufacturing a certain commodity which is to be transported to three warehouses/stock points A, B, C. Availability of commodity at the factory locations (in thousands of tons) P and Q and capacity of the warehouses (in thousands of tons) at A, B and C are given in the following table. Given the following per unit (one ton) cost of transportation from a given location to a desired warehouse in the table, fix a transportation schedule so as to minimize the total cost of transportation.

From \ To	A	B	C	Availability (in thousand tons)
P	20	60	40	10
Q	15	25	80	15
Capacity (in thousand tons)	8	10	7	

Solution : We shall first formulate the given transportation problem as a LPP.

Let x and y denote the amount of commodity to the transported from a location P to the warehouse A and B respectively. Then, from the total availability and capacity constraints given in the hypothesis, we can construct the following table to show the amount of commodity to be transported from the two factory locations to all the warehouses.

Commodity From \ To	A	B	C
P	x	y	$10-(x+y)$
Q	$8-x$	$10-y$	$15-[(8-x)+(10-y)]$ $=x+y-3$

Then we can form the following constraints and non negative restrictions for the given problem

$x \geq 0,\ y \geq 0$: non negative restrictions.

$$\left.\begin{array}{l} 8-x \geq 0 \Rightarrow x \leq 8 \\ 10-y \geq 0 \Rightarrow y \leq 10 \\ 10-(x+y) \geq 0 \Rightarrow x+y \leq 10 \\ x+y-3 \geq 0 \Rightarrow x+y \geq 3 \end{array}\right\} \text{Constraint inequations}$$

The total cost of transporting the output =

$$20x + 60y + 40\,[10 - (x + y)] + 15(8 - x) + 25(10 - y) + 80(x + y - 3)$$
$$= 45x + 75y + 530$$

Thus objective function for the given problem is

Minimize $z = 45x + 75y + 530.$

We shall now draw the graph corresponding to the constraint equations (treating the inequations as equations) to solve the transportation Problem (Fig. 2.12).

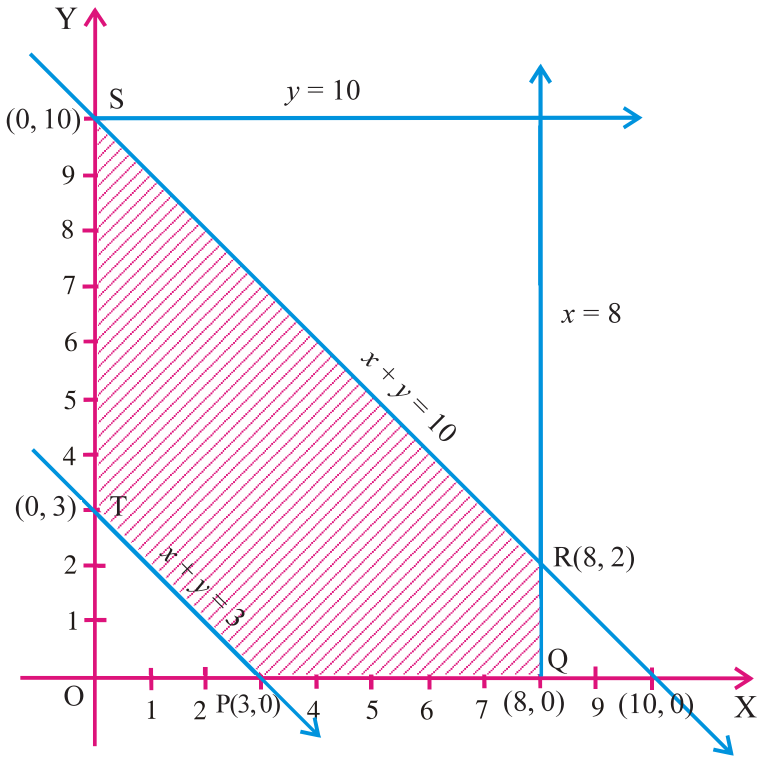

Fig. 2.12

Observe here that the solution set of the constraint is a closed convex polygon PQRST with the coordinates P(3, 0), Q(8, 0), R(8, 2), S(0, 10) and T(0, 3). The value of the objective function at these corner points, is as follows.

P(3, 0) = 135 + 530 = ₹ 665

Q(8, 0) = 360 + 530 = ₹ 890

R(8, 2) = 510 + 530 = ₹ 1040

S(0, 10) = 750 + 530 = ₹ 1280

T(0, 3) = 225 + 530 = ₹ 755.

The minimum value of z is attained at P(3, 0). Hence $x = 3$ and $y = 0$. Hence we have the following transportation schedule of the commodity from the factory locations to the warehouses.

From \ To	A	B	C
	(in thousands of tons)		
P	3	0	7
Q	5	10	0

Exercise 2(a)

1. A shopkeeper sells not more than 30 shirts of each colour. The green ones are sold atleast twice as many as the white ones. If the profit on each of the white one is ₹ 20 and that of the green one is ₹ 25, how many of each kind are to be sold inorder to give him the maximum profit.

2. A sweet shop keeper makes gift packets each weighing 7 kgs, with two special type of sweets A and B. At least 3 kgs of A and no more than 5 kgs of B, should be used in a packet. The shop keeper makes a profit of ₹ 15 on A and ₹ 20 on B per kg. Determine the product mix so as to realise the maximum profit.

3. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains atleast 10 units of vitamin A and 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg. of food is given in the following table :

Food	Vitamin A	Vitamin B	Vitamin C
X	1	2	3
Y	2	2	1

One kg. of food X costs ₹ 16 and one kg.of food Y costs ₹ 20. Find the least cost of the mixture that produces the prescribed diet.

4. An oil company has two stock Points A and B with capacities 7000 L and 4000 L respectively. It has to supply oil to three petrol pumps D, E and F whose requirements are 4500 L, 3000 L and 3500 L respectively. The distances (in km) between the stock points and the petrol pumps is given in the table:

To / From	Distance (in km)	
	A	B
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost of 10 lt of oil per km is ₹ 1, determine the delivery schedule in order that the total transportation cost is a minimum. Find also the minimum transportation cost.

5. A manufacturer makes two types of Toys A and B. He needs 3 machines for this purpose and the time (in minutes) required to process each toy on the machines is given below :

Type of Toys	Processing time on Machines (minutes)		
	I	II	III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of 6 hours per day. If the profit on each toy of type A is ₹ 7.50 and that on each toy of type B is ₹ 5, determine the number of toys of each type he has to manufacture, to get the maximum profit.

6. A dealer sells a desk top model and a laptop model of computers whose cost is ₹ 25000 and ₹ 40000 respectively. His estimate of total monthly demand of computers will not exceed 250 units. His profit on desktop model is ₹ 4500 and on the laptop model is ₹ 5000. Determine the number of units of each type of computers the dealer has to stock to get the maximum profit, if his total investment on the stock does not exceed ₹ 70 lakhs.

Answers

Exercise 2(a)

1. White Shirts 20, green shirts 10 and max profit ₹ 650
2. Each 7 kg packet should contain 3 kgs of type A, 4 kgs of type B sweets.
3. 2 kgs .of food X and 4 kgs. of food Y. Least cost of the mixture is ₹ 112.
4.

From \ To	D	E	F
A	500	3000	3500
B	4000	-	-

Min cost is ₹ 4400.

5. 15 toys of type A, 30 toys of type B.
6. 200 units of desk top model and 50 units of laptop model. Max. profit is ₹ 11,50,000.

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06. Binomial Theorem	
6.1 Binomial theorem for positive integral index	12
6.2 Binomial theorem for rational Index (without proof)	08
6.3 Approximations using Binomial theorem	04
	24
07. Partial Fractions	
7.1 Partial fractions of $f(x) / g(x)$ when $g(x)$ contains non - repeated linear factors	02
7.2 Partial fractions of $f(x) / g(x)$ when $g(x)$ contains repeated and / or non-repeated linear factors	03
7.3 Partial fractions of $f(x) / g(x)$ when $g(x)$ contains irreducible factors	02
	07

PROBABILITY

08. Measures of Dispersion	
8.1 Range	01
8.2 Mean deviation	03
8.3 Variance and standard deviation of ungrouped/ grouped data	07
8.4 Coefficient of variation and analysis of frequency distributions with equal means but different variances	04
	15
09. Probability	
9.1 Random experiments and events	06
9.2 Classical definition of probability, Axiomatic approach and addition theorem of probability	05
9.3 Independent and dependent events conditional probability- multiplication theorem and Baye's theorem	07
	18
10. Random Variables and Probability Distributions	
10.1 Random Variables	04
10.2 Theoretical discrete distributions - Binomial and Poisson Distributions	07
	11
TOTAL	**150**

ADDITIONAL READING MATERIAL

For the benefit of students who want to appear for competitive exams based on COBSE the following topics may be given as Additional Reading Material.

1. **Exponential and Logarithmic Series**

2. **Linear Programming**

BOARD OF INTERMEDIATE EDUCATION, A.P.

Mathematics - IIA

Model Question Paper (w.e.f. 2013-14)

Time: 3 hrs **Max. Marks:75**

Note: This Question paper consists of three sections A, B and C.

SECTION - A

I. Very Short Answer type Questions

(i) Answer all Questions

(ii) Each Question carries 2 marks **$10 \times 2 = 20$**

1. Find the square root of $-5 + 12i$.

2. If $z_1 = -1,\ z_2 = i$ then find $\operatorname{Arg}\left(\frac{z_1}{z_2}\right)$.

3. Find the value of $(1+i)^{16}$.

4. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

5. Find the algebraic equation whose roots are two times the roots of $x^5 - 2x^4 + 3x^3 - 2x^2 + 4x + 3 = 0$.

6. Find the number of ways of arranging the letters of the word "INTERMEDIATE".

7. If ${}^{n}P_{r} = 5040$ and ${}^{n}C_{r} = 210$ find n and r.

8. If $(1 + x + x^2)^n = a_o + a_1 x + a_2 x^2 + ... + a_{2n} x^{2n}$ then find the value of $a_0 + a_2 + a_4 + ... + a_{2n}$.

9. The variance of 20 observations is 5. If each observation is multiplied by 2, then find the new variance of the resulting observations.

10. A poisson variable satisfies $P(x=1) = P(x=2)$. Find $P(X=5)$

SECTION - B

II. Short Answer type Questions

(i) Answer any five Questions

(ii) Each Question carries 4 marks **$5 \times 4 = 20$**

11. If $z = x + iy$ and if the point P in the Argand plane represents z, find the locus of z satisfying the equation $|z - 2 - 3i| = 5$.

12. Find the range of $\dfrac{x+2}{2x^2+3x+6}$.

13. If the letters of the word MASTER are permuted in all possible ways and the words thus formed are arranged in the dictionary order, then find the rank of the word "REMAST".

14. Find the number of ways of selecting a cricket team of 11 players from 7 batsmen and 6 bowlers such that there will be atleast 5 bowlers in the team.

15. Resolve $\dfrac{x^2-3}{(x+2)(x^2+1)}$ into partial fractions.

16. Two persons A and B are rolling a die on the condition that the person who gets 3 will win the game. If A starts the game, then find the probabilities of A and B respectively to win the game.

17. A problem in calculus is given to two students A and B whose chances of solving it are $\dfrac{1}{3}$ and $\dfrac{1}{4}$ respectively. Find the probability of the problem being solved if both of them try independently.

SECTION - C

III. Long Answer type Questions

(i) Answer any five Questions

(ii) Each Question carries 7 marks **$5 \times 7 = 35$**

18. Find all the roots of the equation $x^{11} - x^7 + x^4 - 1 = 0$.

19. Solve : $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.

20. If n is a positive integer and x is any nonzero real number, then prove that

$$C_0 + C_1\frac{x}{2} + C_2.\frac{x^2}{3} + C_3.\frac{x^3}{4} + ... + C_n.\frac{x^n}{n+1} = \frac{(1+x)^{n+1}-1}{(n+1)x}.$$

21. If $x = \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + ...$ then prove that $9x^2 + 24x = 11$.

22. Calculate the variance and standard deviation for the following distribution:

Class	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

23. The probabilities of three events A, B, C are such that P(A) = 0.3, P(B) = 0.4, P(C) = 0.8, $P(A \cap B) = 0.08$, $P(A \cap C) = 0.28$, $P(A \cap B \cap C) = 0.09$, and $P(A \cup B \cup C) \geq 0.75$, show that $P(B \cap C)$ lies in the interval [0.23, 0.48].

24. A random variable x has the following probability distribution.

$X = x_i$	0	1	2	3	4	5	6	7
$P(X = x_i)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Find (i) k (ii) the mean (iii) $P(0 < X < 5)$.